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Correlation Inequalities and the Thermodynamic Limit for Classical and Quantum Continuous Systems II. Bose–Einstein and Fermi–Dirac Statistics

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We study quantum mechanical systems of particles with Bose or Fermi statistics interacting via two-body potentials of positive type in thermal equilibrium. We rewrite partition functions, reduced density matrices (RDMs), and correlation functions in terms of Wiener and Gaussian functional integrals (sine-Gordon transformation). This permits us, e.g., to apply correlation inequalities. Our main results include an analysis of stability versus instability in the grand canonical ensemble and, for charge-conjugation-invariant systems, upper and lower bounds on RDMs, the existence of the thermodynamic limit of pressure, RDMs and correlation functions, an inequality comparing correlations with Fermi statistics to ones with Bose statistics, and inequalities which are important in the study of Bose–Einstein condensation and of superconductivity.

KEY WORDS: Correlation inequalities; classical and quantum continuous systems; positive type potentials; stability; thermodynamic limit.

1. NOTATIONS AND SUMMARY OF RESULTS

1.1. An Outline of the Main Results

In this paper we continue our study, initiated in Ref. 10 (hereafter referred to as I), of classical and quantum mechanical continuous systems in thermal equilibrium. The systems considered here consist of two (or more) species of particles interacting via two-body potentials of positive type, and in many results an exact charge conjugation invariance is required. In I we have

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found new correlation inequalities of the Ginibre type[13] for classical systems and quantum mechanical systems without statistics ("Boltzmann statistics") which are charge conjugation invariant. We applied those correlation inequalities to establish the existence of the thermodynamic limit of (the pressure and) the correlation (RDM) and imaginary time Green's functions (ITGF).

One basic ingredient in the proof of those results, in particular of the correlation inequalities, was the use of a combination of the Feynman–Kac formula with the sine-Gordon (or Siegert) transformation. These technical devices play a decisive role in the present paper as well.

The results of paper I are here extended in the following four directions:

(A) Analysis of stability and instability of quantum mechanical systems in the grand canonical ensemble. Let $\Xi_\Lambda(\beta, z)$ denote the grand canonical partition function in a bounded region Λ (of volume $|\Lambda|$) at inverse temperature β and activity z. By *stability* we mean the inequality

$$\Xi_\Lambda(\beta, z) \leqslant \exp[O(|\Lambda|)]$$

for suitable $\beta > 0$ and $z > 0$. By *instability* we mean, roughly speaking, that for some β, $\Xi_\Lambda(\beta, z)$ has a singularity in z at some finite, positive value z_0, and $\Xi_\Lambda(\beta, z) = +\infty$ for $z > z_0$. See Section 2.

(B) Proof of the existence of the thermodynamic limit of the pressure for stable two-component, charge-conjugation-invariant systems of arbitrary statistics. See Section 2.

Remark. In (A) and (B) the particles may have arbitrary spin, and the results outlined in (A) do not require charge conjugation invariance.

(C) Uniform (in Λ) upper and lower bounds on RDMs and ITGFs for charge conjugation invariant systems with Bose statistics and sufficiently small activity (Fermi statistics also can be treated with our methods, but the resulting bounds are not particularly useful). See Section 5.

(D) Existence of the thermodynamic limit of RDMs and ITGFs of charge-conjugation-invariant systems with Bose statistics (below the breakdown of stability). We also prove a comparison inequality between RDMs with Fermi (resp. Bose) statistics and study the effects of interactions with the electromagnetic field. See Section 5.

In Section 3 we recall and extend the correlation inequalities of I. In Section 4 we give a simple derivation of Ginibre's formulas (see Ref. 12) for the RDMs and ITGFs by using Gaussian functional integrals, as in Ref. 7 and paper I (*sine-Gordon* or *Siegert transformation*), in conjunction with Brownian motion. The sine-Gordon transformation permits a derivation of Ginibre's formulas for the RDMs from elementary, known facts concerning ideal gases of particles in an external (purely imaginary) potential. This gives our elaborations on the sine-Gordon transformation a certain degree of perfection. We also show how to include interactions with the (classical or quantized) electromagnetic field in this formalism.

1.2. Some Notations and Definitions

The physical systems we study consist of two species of quantum mechanical particles of mass m_i and charge q_i, $i = 1, 2$ (several of our results extend, however, to arbitrarily many species of particles of arbitrary masses and charges). These particles interact via two-body potentials. The potential between a particle of charge q at a point $x \in \mathbb{R}^\nu$ and one of charge q' at a point $x' \in \mathbb{R}^\nu$ is given by $qq'V(x, x')$. Generally $\nu = 3$. Henceforth it will be required (unless otherwise stated) that V be of positive type, i.e., that it be the integral kernel of a positive quadratic form on $L^2(\mathbb{R}^\nu)$. Moreover, we shall usually assume that $V(x, x')$ is continuous in x and x' and

$$K \equiv \sup_{x \in \mathbb{R}^\nu} V(x, x) < \infty \tag{1.1}$$

We are primarily interested in translation-invariant potentials,

$$V(x, x') = V(x - x') \tag{1.2}$$

Then V is of positive type iff its Fourier transform $\hat{V}$ is nonnegative.

Condition (1.1) can be relaxed significantly for classical systems[7,10] and quantum mechanical systems with Boltzmann—or Fermi (resp. mixed Bose and Fermi) statistics; see Ref. 17.

First, we consider systems confined to a bounded, open region $\Lambda \subset \mathbb{R}^\nu$. The coordinates of N particles of species 1 are denoted $(x)_N = (x_1,..., x_N)$, $x_j \in \Lambda$, those of M particles of species 2 are $(x')_M = (x_1',..., x_M')$, $x_j' \in \Lambda$, and

$$d(x)_N = \prod_{j=1}^{N} d^\nu x_j, \qquad d(x')_M = \prod_{j=1}^{M} d^\nu x_j' \tag{1.3}$$

The one-particle Hilbert space for a particle of species i is $L^2(\Lambda, d^\nu x) \otimes \mathbb{C}^{2S_i+1}$, where S_i is the spin of the particle; the N-particle Hilbert space $\mathscr{H}_\Lambda^{(N)}$ is given by

$$\mathscr{H}_{i,\Lambda}^{(N)} = (L^2(\Lambda, d^\nu x) \otimes \mathbb{C}^{2S_i+1})^{\otimes_{\epsilon_i} N} \tag{1.4}$$

where ϵ_i is the statistics of those particles; $\epsilon_i = +1$ for Bose statistics, $\epsilon_i = -1$ for Fermi statistics.

The total Hilbert space of N particles of species 1 and M of species 2 is

$$\mathscr{H}_\Lambda^{(N,M)} = \mathscr{H}_{1,\Lambda}^{(N)} \otimes \mathscr{H}_{2,\Lambda}^{(M)} \tag{1.4'}$$

Let $\Delta_j^{(\prime)\Lambda}$ be the Laplacian on $L^2(\Lambda, d^\nu x_j^{(\prime)})$ with zero Dirichlet data at the boundary $\partial\Lambda$ of Λ. The Hamiltonian of the (N, M)-particle system is given by

$$H_\Lambda^{(N,M)} = -\sum_{i=1}^{N} (1/2m_1)\Delta_i^\Lambda - \sum_{j=1}^{M} (1/2m_2)\Delta_j'^\Lambda + U((x)_N, (x')_M) \quad (1.5)$$

where

$$U((x)_N, (x')_M) = \sum_{1 \leqslant i < j \leqslant N} q_1{}^2 V(x_i, x_j) + \sum_{1 \leqslant i < j \leqslant M} q_2{}^2 V(x_i', x_j') + \sum_{i=1}^{N} \sum_{j=1}^{M} q_1 q_2 V(x_i, x_j')$$

For potentials V of interest in nonrelativistic physics [in particular if V satisfies (1.1)], $H_\Lambda^{(N,M)}$ is known to be self-adjoint on a dense domain in

$$L^2(\Lambda, d^\nu x)^{\otimes(N+M)} \otimes \mathbb{C}^{(2S_1+1)N+(2S_2+1)M}$$

thus on $\mathscr{H}_\Lambda^{(N,M)}$, and $\exp[-\beta H_\Lambda^{(N,M)}]$ is trace class, for bounded, open Λ and $\beta > 0$. Let $r = (x, \mu)$ be the configuration-space point of a particle, where μ labels one component of the spin, and $(r)_N = (r_1, \ldots, r_N)$. Let $\psi_\Lambda{}^\beta((r)_N(r')_M; (\bar{r})_N(\bar{r}')_M)$ be the integral kernel of $\exp[-\beta H_\Lambda^{(N,M)}]$ without statistics. If V is continuous and $\beta > 0$, $\psi_\Lambda{}^\beta$ is well-defined, positive, and continuous in its arguments.

We define

$$\int_{\Lambda^N} d(r)_N \text{---} = \sum_{\mu_1} \cdots \sum_{\mu_N} \int_{\Lambda^N} d(x)_N \text{---}$$

The grand canonical partition function $\Xi_\Lambda(\beta, z)$ is then defined by

$$\begin{aligned}\Xi_\Lambda(\beta, z_1, z_2) &= \sum_{N,M=0}^{\infty} z_1{}^N z_2{}^M \operatorname{Tr}_{\mathscr{H}_\Lambda^{(N,M)}}(\exp[-\beta H_\Lambda^{(N,M)}]) \\ &= \sum_{N=0}^{\infty} \sum_{M=0}^{\infty} \frac{z_1{}^N}{N!} \frac{z_2{}^M}{M!} \int_{\Lambda^{N+M}} d(r)_N \, d(r')_M \\ &\quad \times \sum_{\substack{\pi_1 \in S_N \\ \pi_2 \in S_M}} \epsilon_1^{\sigma(\pi_1)} \epsilon_2^{\sigma(\pi_2)} \psi_\Lambda{}^\beta((r)_N(r')_M; \pi_1(r)_N \pi_2(r')_M) \end{aligned} \quad (1.6)$$

where S_N is the group of permutations of N elements, $|\epsilon_1| = |\epsilon_2| = 1$, and $\sigma(\pi)$ is the signature of π, $\pi(r)_N = (r_{\pi(1)}, \ldots, r_{\pi(N)})$.

The term corresponding to $N = M = 0$ is $\equiv 1$. (In the case of Boltzmann statistics, the sum over permutations is absent. Since the Hamiltonians considered here are spin-independent, we could take a partial trace over all spin degrees of freedom. Then higher dimensional representations of the permutation group on x-space wave functions appear.) The pressure of these systems is given by

$$p_\Lambda(\beta, z_1, z_2) = (1/|\Lambda|) \log \Xi_\Lambda(\beta, z_1, z_2) \tag{1.7}$$

and the RDMs by

$$\begin{aligned}
&\rho_\Lambda(\beta, z; (r)_N(r')_M; (\bar{r})_N(\bar{r}')_M) \\
&= \Xi_\Lambda(\beta, z)^{-1} \sum_{N',M'=0}^{\infty} \frac{z_1^{N+N'} z_2^{M+M'}}{N'!\, M'!} \\
&\times \int_{\Lambda^{N'+M'}} d(u)_{N'}\, d(u')_{M'} \sum_{\substack{\pi_1 \in S_{N+N'} \\ \pi_2 \in S_{M+M'}}} \epsilon_1^{\sigma(\pi_1)} \epsilon_2^{\sigma(\pi_2)} \\
&\times \psi_\Lambda{}^\beta((r)_N(u)_{N'}(r')_M(u')_{M'}; \pi_1((\bar{r})_N(u)_{N'})\pi_2((\bar{r}')(u')_{M'}))
\end{aligned} \tag{1.8}$$

The definition of ITGFs is more complicated; see Ref. 10, Appendix 1.

Definition 1.1. A system is called charge conjugation invariant iff $m_1 = m_2 \equiv m$, $S_1 = S_2$, $\epsilon_1 = \epsilon_2 \equiv \epsilon$, $z_1 = z_2 \equiv z$, and $q_1 \equiv q = -q_2$.

Our main mathematical tools for the analysis of the systems introduced here, in particular of $\Xi_\Lambda(\beta, z)$, $p_\Lambda(\beta, z)$, and $\rho_\Lambda(\beta, z; -)$, appear in Sections 3 and 4.

1.3. Statement of the Main Theorems

In Section 2 we prove, using an idea of Griffiths,[14] the following result.

Theorem A. For stable, charge-conjugation-invariant systems with arbitrary statistics and translation-invariant potential V, the thermodynamic limit

$$p(\beta, z) = \lim_{\Lambda \nearrow \mathbb{R}^\nu} p_\Lambda(\beta, z)$$

exists and is independent of the sequence Λ (only assumed to be increasing). The limit $p(\beta, z)$ has the usual convexity properties.

In Section 5 we extend the results of I by proving the following:

Theorem B. *For stable, charge-conjugation-invariant systems of bosons* ($\epsilon = 1$) *the thermodynamic limit of the RDMs*

$$\rho(\beta, z; (x)_N(x')_M; (y)_N(y')_M) = \lim_{\Lambda \nearrow \mathbb{R}^\nu} \rho_\Lambda(\beta, z; (x)_N(x')_M; (y)_N(y')_M)$$

exists for all N, $M = 0, 1, 2,\ldots$. It is monotone increasing in z and bounded above by

$$\left[\sum_{\pi \in S_N} \prod_{j=1}^{N} \tilde{\rho}(x_j, y_{\pi(j)})\right]\left[\sum_{\pi' \in S_M} \prod_{j=1}^{M} \tilde{\rho}(x_j', y_{\pi'(j)}')\right]$$

where

$$\tilde{\rho}(x, y) = \sum_{j=1}^{\infty} \frac{1}{\lambda^{\nu}} \frac{\tilde{z}^j}{j^{\nu/2}} \exp\left[-\frac{|x-y|^2}{2j\beta} \right]$$

and $\lambda = (2\pi m/\beta)^{1/2}$, for $\tilde{z} = z \exp(\beta K/2) < 1$.

Remarks. 1. We show in Section 2.2 that the restriction $\tilde{z} < 1$ in Theorem B cannot be relaxed by much because for large z, Bose–Einstein condensation destroys stability.

2. The proof of Theorem B is based on a combined use of Brownian motion, the sine-Gordon transformation (Section 4), and correlation inequalities (Section 3). Whereas the first two techniques can be used to analyze very general systems of particles of arbitrary spin and statistics, it appears that the correlation inequalities only hold for charge-conjugation-invariant systems with Bose statistics. Theorem B can be extended to bosons with integral spin. This is a straightforward generalization of the techniques developed in Sections 3–5 which we do not elaborate on; but see Section 4.

3. Theorems A and B can be generalized to the case where the particles carry electric charge through which they are coupled to the quantized radiation field by minimal substitution

$$\Delta_j^{\Lambda} \to \Delta_{\mathbf{A},j}^{\Lambda} = \{[\nabla \pm ie\mathbf{A}_{\kappa}(x_j)]^*[\nabla_j \pm ie\mathbf{A}_{\kappa}(x_j)]\}^{\Lambda} \tag{1.9}$$

Here $\mathbf{A}$ is the quantized vector potential, and κ is an ultraviolet cutoff with the effect that the two-point function of $\mathbf{A}_{\kappa}(0)$ is finite. We shall discuss the generalization of Theorem B to such systems in some detail. Moreover, we shall show that the RDMs of charge-conjugation-invariant Bose systems with $e \neq 0$ are bounded above by the ones with $e = 0$ (Section 5).

4. Following Appendix 1 of Ref. 10, one can extend the results of Theorem B to the ITGFs. That permits the reconstruction of a unique KMS state and of the dynamics in the corresponding KMS representation, in the thermodynamic limit.[23]

5. Existence theorems for the RDMs and ITGFs of quantum mechanical systems in the grand canonical ensemble have previously been obtained for various classes of short-range potentials in Ref. 12 and for nonrelativistic matter with Coulomb replaced by Yukawa potentials in Ref. 3. The methods used there only work in the dilute regime (small β and z) and for short-range potentials. In comparison, our methods work for arbitrary values of β and an optimal range of z and do not impose restrictions on the range of the potentials. Moreover, the quantized radiation field can be included in our treatment. However, our assumptions of Bose (or Boltzmann) statistics and strict charge conjugation invariance are physically awkward.

6. Among our further results are (see Section 5) (a) an inequality saying

that for fixed parameters and given potential, the absolute values of the RDMs with Fermi statistics are bounded above, in configuration space, by ones with Bose statistics; (b) lower bounds for the RDMs of charge-conjugation-invariant Bose gases which diverge if z is large enough and

$$|V(x)| \leqslant O(|x|^{-1-\epsilon}) \qquad \text{as } |x| \to \infty \qquad \text{for some } \epsilon > 0$$

Our results suggest that such systems are likely to exhibit *Bose–Einstein condensation*, and that one ought to be able to even prove this rigorously; (c) various (diamagnetic and other) inequalities for the partition functions and the RDMs of systems coupled to the electromagnetic field that might be of interest in the theory of superconductivity.

2. STABILITY AND INSTABILITY IN THE GRAND CANONICAL ENSEMBLE. THERMODYNAMIC LIMIT OF THE PRESSURE

2.1. Stability in the Grand Canonical Ensemble

Consider the Hamiltonian $H_\Lambda^{(N,M)}$ on the Hilbert space $\mathscr{H}_\Lambda^{(N,M)}$ defined in Section 1.2, Eqs. (1.5) and (1.4), respectively. The basic assumption is that $\epsilon_1 = -1$ (i.e., the first species of particles consists of fermions) and that the interaction potential V is chosen such that, for a given choice of ϵ_2, the system is H-stable in the sense that for some finite constant B and arbitrary Λ

$$H_\Lambda^{(M,N)} > \tfrac{1}{2} T_\Lambda^{(M,N)} - B(M+N) \tag{2.1}$$

as a quadratic form, for arbitrary M and N; here $T_\Lambda^{(M,N)}$ is the kinetic energy operator, i.e.,

$$T_\Lambda^{(M,N)} = -\sum_{i=1}^{M} (1/2m_1)\Delta_i^\Lambda - \sum_{j=1}^{N} (1/2m_2)\Delta_j'^\Lambda \tag{2.2}$$

See (1.5).

Definition. A region $\Lambda \subset \mathbb{R}^\nu$ is called regular iff $\operatorname{diam} \Lambda \leqslant \alpha|\Lambda|^{1/\nu}$ for some finite α.

Theorem 2.1. Let $\epsilon_1 = -1$, $\epsilon_2 = \pm 1$, and $q_2 \neq 0$. Assume that the potential V is of the form

$$V(x, y) = V_1(x, y) + V_2(x - y) \tag{2.3}$$

such that (2.1) holds for $V = V_1$, $V_2 = 0$, and V_2 is a function whose Fourier transform $\hat{V}_2$, is nonnegative and continuous with

$$\hat{V}_2(0) > 0$$

Let m_1, m_2 be positive and z_1, z_2 finite. Then there exists a constant $c = c(\beta, z_1, z_2)$, finite for all $\beta > 0$, such that

$$\Xi_\Lambda(\beta, z_1, z_2) < \exp[c|\Lambda|] \qquad (\Xi\text{-stability}) \tag{2.4}$$

for arbitrary, regular regions Λ.

A possibly novel, simple *proof* of Theorem 2.1 is given in Appendix A.

Remarks. 1. The result in Section 2.2 shows that it is important to assume that one species of particles consists of fermions.

2. Theorem 2.1 has an obvious generalization to systems of arbitrarily many species of particles including fermions.

3. As an application, consider the three-dimensional, nonrelativistic matter system, with V, e.g., the Coulomb potential. We decompose V into two parts,

$$V = V_1 + V_2$$

with

$$V_1(x) = (1/4\pi|x|)e^{-\mu|x|}, \qquad V_2(x) = (1/4\pi|x|)(1 - e^{-\mu|x|})$$

for some $\mu > 0$. We assume that one species of particles is fermions. Then all hypotheses of Theorem 2.1 are valid. Thus, the grand canonical partition function of the matter system satisfies inequality (2.4), i.e., the system is "Ξ-stable."

We have recovered here a result of Lieb and Lebowitz.[17]

2.2. Instability in the Grand Canonical Ensemble

In this section we study a two-component, pure-boson system with dynamics given by the Hamiltonians $H_\Lambda^{(N,M)}$, N, $M = 0, 1, 2,\ldots$, but in contrast to Section 2.1, we assume $\epsilon_1 = \epsilon_2 = 1$. The masses of the particles in the two species are m_1, m_2, their charges are q_1, q_2 with $q_1 > 0$, $q_2 < 0$, and their activities are z_1, z_2, respectively. We set

$$m = \min\{m_1, m_2\}, \qquad z = \tfrac{1}{2}\min\{z_1, z_2\}\min\{1 - q_1/q_2, 1 - q_2/q_1\} \tag{2.5}$$

Moreover, we define $\Xi_\Lambda{}^0(\beta, z)$ to be the partition function of an ideal, one-component Bose gas of particles with mass m. In Appendix B we prove the following:

Theorem 2.2. Consider the system described above, with $\epsilon_1 = \epsilon_2 = 1$ and m and z as defined in (2.5). Then

$$\Xi_\Lambda(\beta, z_1, z_2) \geqslant \Xi_\Lambda{}^0(\beta, z) \tag{2.6}$$

Remarks. 1. It is well known that, for arbitrary $z > 1$, there exists $\Lambda_0(z)$ such that $\Xi_\Lambda{}^0(\beta, z)$ is divergent for $\Lambda \supseteq \Lambda_0(z)$. Thus, Theorem 2.2 says that for z_1 and z_2 large enough depending on q_1 and q_2, *the two-component Bose systems considered here are not* Ξ*-stable.*

2. In Section 5 we show that for two-component, charge-conjugation-invariant Bose systems with pair potential decaying like $|x|^{-1-\epsilon}$, $\epsilon > 0$, the RDMs diverge for z large enough.

2.3. The Thermodynamic Limit of the Pressure of Charge-Conjugation-Invariant Systems

In this section we study general charge-conjugation-invariant systems of arbitrarily many species of particles with arbitrary spin and statistics. All that is important is (i) strict charge conjugation invariance, (ii) Ξ-stability.

Under the above hypotheses we prove the existence of the thermodynamic limit of the pressure (grand canonical ensemble), using an idea due to Griffiths.[14] In order to economize on notations, we restrict our attention to two-component, quantum mechanical systems, but our methods work in the general case as well. Moreover, they are applicable to charge-conjugation-invariant, classical systems and, after some modifications, to one-component systems with nonnegative potentials. They are, however, too simple-minded to permit to study the dependence of the thermodynamic limit on boundary conditions.

The main result of this section is as follows:

Theorem 2.3. Consider the pressure $p_\Lambda(\beta, z) = p_\Lambda(\beta, z, z)$ defined in (1.7) of a system with dynamics given by the Hamiltonians (1.5). Suppose that the system is Ξ-stable in the sense of inequality (2.4), and charge conjugation invariant in the sense of Definition 1.1. Then

$$p(\beta, z) \equiv \lim_{\Lambda \uparrow \mathbb{R}^\nu} p_\Lambda(\beta, z) \tag{2.7}$$

exists and has the usual convexity properties (provided $\Lambda \uparrow \mathbb{R}^\nu$ in the sense of Van Hove or Fisher[22]).

Proof. Let Λ_1, Λ_2 be bounded, open subsets of $\mathbb{R}^\nu$ with $\Lambda_1 \cap \Lambda_2 = \varnothing$. Let $\Xi(\beta, z) \equiv \Xi_\Lambda(\beta, z, z)$ be the grand canonical partition function. By general arguments[22] it is enough to show that

$$\Xi_{\Lambda_1 \cup \Lambda_2}(\beta, z) \geqslant \Xi_{\Lambda_1}(\beta, z)\Xi_{\Lambda_2}(\beta, z) \tag{2.8}$$

provided Ξ-stability holds.

We introduce the Hilbert space

$$\mathscr{H}_\Lambda = \bigoplus_{N,M=0}^{\infty} \mathscr{H}_\Lambda^{(N,M)} \tag{2.9}$$

where $\mathscr{H}_\Lambda^{(N,M)}$ has been defined in (1.4). We set $\mu = -\beta^{-1} \log z$ and define the Hamiltonian H_Λ on $\mathscr{H}_\Lambda$ by

$$H_\Lambda = \bigoplus_{N,M=0}^{\infty} [H_\Lambda^{(N,M)} + \mu(N+M)\mathbb{1}]|_{\mathscr{H}_\Lambda^{(N,M)}} \tag{2.10}$$

with $H_\Lambda^{(N,M)}$ the Hamiltonian defined in (1.5). One convinces oneself by direct calculation that

$$\Xi_\Lambda(\beta, z) = \mathrm{Tr}_{\mathscr{H}_\Lambda}(e^{-\beta H_\Lambda}) \tag{2.11}$$

Next, let $T_\Lambda^{(N,M)}$ be the kinetic energy operator introduced in (2.2), and

$$T_\Lambda = \bigoplus_{N,M=0}^{\infty} T_\Lambda^{(N,M)}$$

Given two bounded, open subsets Λ_1, Λ_2, we define

$$\begin{aligned} W_{\Lambda_1,\Lambda_2} = \bigoplus_{N_1,M_1=0}^{\infty} \bigoplus_{N_2,M_2=0}^{\infty} \{&U((x)_{N_1}(x')_{M_1}, (y)_{N_2}(y')_{M_2}) \\ &- U((x)_{N_1}, (x')_{M_1}) - U((y)_{N_2}, (y')_{M_2})\} \end{aligned} \tag{2.12}$$

with the convention that the x and x' coordinates are in Λ_1, whereas the y and y' coordinates are in Λ_2. Clearly, W_{Λ_1,Λ_2} is the interaction energy between the system confined to Λ_1 and the one confined to Λ_2.

Since 0-Dirichlet data are imposed on T_Λ [see (1.5)],

$$T_{\Lambda_1 \cup \Lambda_2} \leqslant T_{\Lambda_1} + T_{\Lambda_2} \tag{2.13}$$

(This follows from $-\Delta^{\Lambda_1 \cup \Lambda_2} \leqslant -\Delta^{\Lambda_1} - \Delta^{\Lambda_2}$, a well-known inequality.) By (2.13) and definitions (2.10) and (2.12),

$$H_{\Lambda_1 \cup \Lambda_2} \leqslant H_{\Lambda_1} + H_{\Lambda_2} + W_{\Lambda_1,\Lambda_2} \tag{2.14}$$

Therefore

$$\mathrm{Tr}_{\mathscr{H}_\Lambda}[\exp(-\beta H_{\Lambda_1 \cup \Lambda_2})] \geqslant \mathrm{Tr}_{\mathscr{H}_\Lambda}\{\exp[-\beta(H_{\Lambda_1} + H_{\Lambda_2} + W_{\Lambda_1,\Lambda_2})]\} \tag{2.15}$$

Let $\rho_{\Lambda_1,\Lambda_2}$ be the state given by

$$\mathrm{Tr}_{\mathscr{H}_\Lambda}\{\exp[-\beta(H_{\Lambda_1} + H_{\Lambda_2})]\}^{-1} \mathrm{Tr}_{\mathscr{H}_\Lambda}\{-\exp[-\beta(H_{\Lambda_1} + H_{\Lambda_2})]\}$$

The Peierls–Bogoliubov inequality now gives

$$\begin{aligned} &\mathrm{Tr}_{\mathscr{H}_\Lambda}\{\exp[-\beta(H_{\Lambda_1} + H_{\Lambda_2} + W_{\Lambda_1,\Lambda_2})]\} \\ &\quad \geqslant \mathrm{Tr}_{\mathscr{H}_\Lambda}\{\exp[-\beta(H_{\Lambda_1} + H_{\Lambda_2})]\} \exp[-\beta\rho_{\Lambda_1,\Lambda_2}(W_{\Lambda_1,\Lambda_2})] \end{aligned} \tag{2.16}$$

Next, we note that if V_1 and V_2 are two closed, orthogonal subspaces of a Hilbert space, then

$$(V_1 \oplus V_2)^{\otimes_\epsilon n} = \bigoplus_{k=0}^{n} V_1^{\otimes_\epsilon k} \otimes V_2^{\otimes_\epsilon (n-k)}$$

so that

$$\bigoplus_{n=0}^{\infty} (V_1 \oplus V_2)^{\otimes_\epsilon n} = \left(\bigoplus_{m=0}^{\infty} V_1^{\otimes_\epsilon m}\right) \otimes \left(\bigoplus_{l=0}^{\infty} V_2^{\otimes_\epsilon l}\right)$$

If we set $V_i = L^2(\Lambda_i, d^\nu x) \otimes \mathbb{C}^{2S+1}$, $i = 1, 2$, and recall (1.4′) and (2.9) we obtain

$$\mathscr{H}_{\Lambda_1 \cup \Lambda_2} = \mathscr{H}_{\Lambda_1} \otimes \mathscr{H}_{\Lambda_2} \tag{2.17}$$

Furthermore,

$$\exp[-\beta(H_{\Lambda_1} + H_{\Lambda_2})]|_{\mathscr{H}_{\Lambda_1 \cup \Lambda_2}} = \exp(-\beta H_{\Lambda_1})|_{\mathscr{H}_{\Lambda_1}} \otimes \exp(-\beta H_{\Lambda_2})|_{\mathscr{H}_{\Lambda_2}} \tag{2.18}$$

Let ρ_{Λ_i} be the state given by

$$\mathrm{Tr}_{\mathscr{H}_{\Lambda_i}}[\exp(-\beta H_{\Lambda_i})]^{-1}\, \mathrm{Tr}[-\exp(-\beta H_{\Lambda_i})], \qquad i = 1, 2$$

By (2.17) and (2.18), $\rho_{\Lambda_1, \Lambda_2} = \rho_{\Lambda_1} \otimes \rho_{\Lambda_2}$.

Using in addition (2.16), we arrive at

$$\begin{aligned} &\mathrm{Tr}_{\mathscr{H}_\Lambda}\{\exp[-\beta(H_{\Lambda_1} + H_{\Lambda_2} + W_{\Lambda_1,\Lambda_2})]\} \\ &\quad \geqslant \mathrm{Tr}_{\mathscr{H}_{\Lambda_1}}[\exp(-\beta H_{\Lambda_1})]\, \mathrm{Tr}_{\mathscr{H}_{\Lambda_2}}[\exp(-\beta H_{\Lambda_2})] \\ &\quad\quad \times \exp[-\beta \rho_{\Lambda_1} \otimes \rho_{\Lambda_2}(W_{\Lambda_1,\Lambda_2})] \end{aligned} \tag{2.19}$$

Using the product structure of $\rho_{\Lambda_1} \otimes \rho_{\Lambda_2}$, (2.12), and charge conjugation invariance, one sees immediately that

$$\rho_{\Lambda_1} \otimes \rho_{\Lambda_2}(W_{\Lambda_1,\Lambda_2}) = 0 \tag{2.20}$$

Clearly, (2.15), (2.19), and (2.20) give

$$\mathrm{Tr}_{\mathscr{H}_{\Lambda_1 \cup \Lambda_2}}[\exp(-\beta H_{\Lambda_1 \cup \Lambda_2})] \geqslant \mathrm{Tr}_{\mathscr{H}_{\Lambda_1}}[\exp(-\beta H_{\Lambda_1})]\, \mathrm{Tr}_{\mathscr{H}_{\Lambda_2}}[\exp(-\beta H_{\Lambda_2})]$$

which by (2.11) completes the proof.

Remarks. 1. If one replaces traces by integrals and the Peierls–Bogoliubov inequality by Jensen's inequality, the above proof yields the existence of the thermodynamic limit in *classical*, charge-conjugation-invariant systems; see Ref. 14 and I.

2. Consider a system consisting of only one kind of particle interacting via nonnegative two-body potentials, $\not\equiv 0$. In the definition of $\mathrm{T}_\Lambda^{(N,M)}$ and T_Λ replace Δ^Λ by $\bar{\Delta}^\Lambda$, defined to be the Laplacian with Neumann boundary conditions.

Then

$$T_{\Lambda_1 \cup \Lambda_2} \geqslant T_{\Lambda_1} + T_{\Lambda_2}, \qquad H_{\Lambda_1 \cup \Lambda_2} \geqslant H_{\Lambda_1} + H_{\Lambda_2} + W_{\Lambda_1,\Lambda_2}$$

and $\rho_{\Lambda_1} \otimes \rho_{\Lambda_2}(W_{\Lambda_1,\Lambda_2}) \geqslant 0$. Thus

$$1 \leqslant \Xi_{\Lambda_1 \cup \Lambda_2} \leqslant \Xi_{\Lambda_1}\Xi_{\Lambda_2}$$

which also implies existence of the thermodynamic limit.

3. The strength of the arguments used in the proof of Theorem 2.3 and Remarks 1 and 2 is that they do not impose restrictions on the range of the potentials. Their drawback is that they do not supply detailed information on the properties of the limit $p(\beta, z)$, such as dependence on boundary conditions.[24]

3. CORRELATION INEQUALITIES

First we recall some of the correlation inequalities of I slightly generalized so as to be applicable in our proof of Theorem B, Section 1, which we give in Section 5. Subsequently we establish some new inequalities related to the ones in Refs. 16 and 18, which we shall use to compare the correlation functions of systems in a magnetic field to those of systems without magnetic fields; see Section 5. Let $\mathscr{H}$ be a real Hilbert space, and let C be a (bounded), positive, quadratic form on $\mathscr{H}$. Let ϕ be the Gaussian process indexed by $\mathscr{H}$ with mean 0 and covariance C. The associated Gaussian measure is denoted $d\mu_C(\phi)$; see Ref. 20. Let $(X_j, S_j), j = 1, 2, 3,\ldots$, be a family of measure spaces, and $\{\rho\} = \{d\rho_j\}_{j=1}^{\infty}$ a sequence of measures with the property that $d\rho_j$ is a finite, positive measure on (X_j, S_j), for all j. Let $l^{(j)}, j = 1, 2,\ldots$, be a family of measurable mappings from X_j to $\mathscr{H}$, i.e.,

$$l^{(j)}: \quad x \in X_j \to l_x^{(j)} \in \mathscr{H} \tag{3.1}$$

such that

$$\sum_{j=1}^{\infty} \int_{X_j} d\rho_j(x) \exp[\tfrac{1}{2}C(l_x^{(j)}, l_x^{(j)})] < \infty \tag{3.2}$$

Following the notations of I, Section 2, we define

$$\mathbb{C}(\{\rho\}, \phi) = \sum_{j=1}^{\infty} \int_{X_j} d\rho_j(x) \cos \phi(l_x^{(j)}) \tag{3.3}$$

We introduce a partition function $\Xi(C, \{\rho\})$ by

$$\Xi(C, \{\rho\}) \equiv \Xi(\{\rho\}) = \int d\mu_C(\phi) \exp \mathbb{C}(\{\rho\}, \phi) \tag{3.4}$$

For $F \in L^1(d\mu_C)$ we define

$$\langle F \rangle_{C,\{\rho\}} \equiv \langle F \rangle_{\{\rho\}} = \Xi(\{\rho\})^{-1} \int d\mu_C(\phi)\, F(\phi) \exp \mathbb{C}(\{\rho\}, \phi) \tag{3.5}$$

In the following, $m, n, l, g, \ldots$ denote vectors in $\mathscr{H}$.

Theorem 3.1.

(i) $$\left\langle \prod_{j=1}^{k} \cos \phi(m_j) \right\rangle_{\{\rho\}} > 0$$

(ii) $$\left\langle \prod_{j=1}^{k} \cos \phi(m_j);\ \prod_{i=1}^{r} \cos \phi(n_i) \right\rangle_{\{\rho\}} \equiv \left\langle \prod_{j=1}^{k} \cos \phi(m_j) \prod_{i=1}^{r} \cos \phi(n_i) \right\rangle_{\{\rho\}} - \left\langle \prod_{j=1}^{k} \cos \phi(m_j) \right\rangle_{\{\rho\}} \left\langle \prod_{i=1}^{r} \cos \phi(n_i) \right\rangle_{\{\rho\}} \geqslant 0$$

(iii) $$\left\langle e^{\phi(l)};\ \prod_{i=1}^{r} \cos \phi(n_i) \right\rangle_{\{\rho\}} \leqslant 0$$

Remark. For $\{\rho\} = \{\rho_1, 0, 0, \ldots\}$ Theorem 3.1 is contained in Theorem 3.1 of paper I. See also Ref. 21. The proof of the present generalization is a trivial adaptation of that of Theorem 3.1 of I, which we do not wish to present here. We also recall that

$$\begin{array}{ll} \langle \cos \phi(m) \rangle_{C,\{\rho\}} & \text{is decreasing in } C \\ \langle e^{\phi(l)} \rangle_{C,\{\rho\}} & \text{is increasing in } C \end{array} \tag{3.6}$$

where the order relation for C is the one of quadratic forms. See I, Corollary 3.2.

Let $f^{(j)}$ be a bounded, real-valued function on X_j, $j = 1, 2, 3, \ldots$, and set

$$\mathbb{C}(\{\rho\}, \{f\}, \phi) = \sum_{j=1}^{\infty} \int_{X_j} d\rho_j(x) \cos[\phi(l_x^{(j)}) + f_x^{(j)}] \tag{3.7}$$

Let $\Xi(C, \{\rho\}, \{f\})$ and $\langle - \rangle_{C,\{\rho\},\{f\}}$ be given by (3.4) and (3.5), respectively, but with $\mathbb{C}(\{\rho\}, \phi)$ replaced by $C(\{\rho\}, \{f\}, \phi)$.

Theorem 3.2. Let $\{f^{(j)}\}_{j=1}^{\infty}$ and $\langle - \rangle_{\{\rho\},\{f\}} \equiv \langle - \rangle_{C,\{\rho\},\{f\}}$ be as above. Suppose $d\rho_j \geqslant d|\rho_j'|$ for all $j = 1, 2, 3, \ldots$. Let α, β be real numbers and m, n

vectors in $\mathscr{H}$. Then

$$\begin{aligned}\langle\cos\phi(m)\cos\phi(n)\rangle_{\{\rho\}} &- \langle\cos[\phi(m)+\alpha]\cos[\phi(n)+\beta]\rangle_{\{\rho'\},\{f\}}\\ &\geqslant |\langle\cos\phi(m)\rangle_{\{\rho\}}\langle\cos[\phi(n)+\beta]\rangle_{\{\rho'\},\{f\}}\\ &\quad - \langle\cos[\phi(m)+\alpha]\rangle_{\{\rho'\},\{f\}}\langle\cos\phi(n)\rangle_{\{\rho\}}|\end{aligned}$$

Remarks. 1. Using the identity

$$\prod_{j=1}^{k}\cos\alpha_j = (\tfrac{1}{2})^k\sum_{\{e_j\}}\cos\left(\sum_{j=1}^{k} e_j\alpha_j\right) \tag{3.8}$$

with $e_j = \pm 1, j = 1, 2,..., k$, one obtains trivial generalizations of Theorem 3.2.

2. As a special case of Theorem 3.2, we note that

$$\langle\cos\phi(m)\rangle_{\{\rho\}} \geqslant |\langle\cos[\phi(m)+\alpha]\rangle_{\{\rho'\},\{f\}}|$$

This inequality enables us to compare correlation functions of systems with Bose (resp. Fermi) statistics, with or without couplings to an electromagnetic vector potential. See Section 5.

3. Theorem 3.2 is a variant of recent inequalities due to Lebowitz[16] and extended by Messager *et al.*[18]

Outline of Proof (see also I and Ref. 18). Let ϕ_1, ϕ_2 be two independent Gaussian processes with mean 0 and covariance C. Then

$$\begin{aligned}&\langle\cos\phi(m)\cos\phi(n)\rangle_{\{\rho\}} - \langle\cos\phi(m+\alpha)\cos\phi(n+\beta)\rangle_{\{\rho'\},\{f\}}\\ &\qquad \pm \{\langle\cos\phi(m)\rangle_{\{\rho\}}\langle\cos[\phi(n)+\beta]\rangle_{\{\rho'\},\{f\}}\\ &\qquad - \langle\cos[\phi(m)+\alpha]\rangle_{\{\rho'\},\{f\}}\langle\cos\phi(n)\rangle_{\{\rho\}}\}\\ &\quad = \Xi(\{\rho\})^{-1}\Xi(\{\rho'\},\{f\})^{-1}\int d\mu_C(\phi_1)\,d\mu_C(\phi_2)\\ &\qquad \times\{\cos\phi_1(m)\mp\cos[\phi_2(m)+\alpha]\}\{\cos\phi_1(n)\pm\cos[\phi_2(n)+\beta]\}\\ &\qquad \times\exp\left(\sum_{j=1}^{\infty}\int_{X_j}(d\rho_j+d\rho_j')(x)\{\cos[\phi_1(l_x^{(j)})]+\cos[\phi_2(l_x^{(j)})+f_x^{(j)}]\}\right.\\ &\qquad \left. + (d\rho_j-d\rho_j')(x)\{\cos[\phi_1(l_x^{(j)})]-\cos[\phi_2(l_x^{(j)})+f_x^{(j)}]\}\right)\end{aligned} \tag{3.9}$$

Since the partition functions are positive, it suffices to show that the functional integral on the rhs of (3.9) is nonnegative. We define

$$\left.\begin{aligned}\Psi &= (1/\sqrt{2})(\phi_1+\phi_2)\\ \chi &= (1/\sqrt{2})(-\phi_1+\phi_2)\end{aligned}\right\}\Leftrightarrow\left\{\begin{aligned}\phi_1 &= (1/\sqrt{2})(\Psi-\chi)\\ \phi_2 &= (1/\sqrt{2})(\Psi+\chi)\end{aligned}\right.$$

This transformation is orthogonal in (ϕ_1, ϕ_2) space. Thus

$$d\mu_C(\phi_1)\, d\mu_C(\phi_2) = d\mu_C(\Psi)\, d\mu_C(\chi)$$

(see I). Moreover,

$$\cos \phi_1(m) + \cos[\phi_2(m) + \alpha] = 2 \cos\left\{\frac{1}{\sqrt{2}}\left[\Psi(m) + \frac{\alpha}{2}\right]\right\} \cos\left\{\frac{1}{\sqrt{2}}\left[\chi(m) + \frac{\alpha}{2}\right]\right\}$$

$$\cos \phi_1(m) - \cos[\phi_2(m) + \alpha] = 2 \sin\left\{\frac{1}{\sqrt{2}}\left[\Psi(m) + \frac{\alpha}{2}\right]\right\} \sin\left\{\frac{1}{\sqrt{2}}\left[\chi(m) + \frac{\alpha}{2}\right]\right\}$$

Also, since $d\rho_j$ and $d\rho_j'$ are real measures with $d\rho_j \geqslant d|\rho_j'|$, we have

$$d\rho_j + d\rho_j' \geqslant 0, \qquad d\rho_j - d\rho_j' \geqslant 0, \qquad \text{for all } j \tag{3.10}$$

Inserting all these identities into the functional integral on the rhs of (3.9), expanding then the exponential, and taking into account inequalities (3.10), we see that the functional integral on the rhs of (3.9) can be written as a sum of terms of the form

$$\int d\mu_C(\Psi)\, d\mu_C(\chi)\, F(\Psi)F(\chi) = \left[\int d\mu_C(\Psi)\, F(\Psi)\right]^2$$

with F real-valued. Thus it is nonnegative. ■

We conclude Section 3 by sketching a simple generalization of Theorem 3.1 which is useful for analyzing Bose systems coupled to the quantized radiation field. Let $\mathbb{C}(\{\rho\}, \{f\}, \phi)$ be as in (3.7). We now suppose that the phases $f^{(j)}$ are linear functions of a Gaussian random field A with Gaussian distribution $d\mu(A)$, i.e., $f_x^{(j)} = A(h_x^{(j)})$, for some $\mathscr{H}$-valued functions $h_x^{(j)}, j = 1,..., \infty$. Let

$$\Xi(\{\rho\}) = \int d\mu_C(\phi)\, d\mu(A) \exp[\mathbb{C}(\{\rho\}, \{A(h)\}, \phi] \tag{3.11}$$

$$\langle - \rangle_{\{\rho\}} = \Xi(\{\rho\})^{-1} \int d\mu_C(\phi)\, d\mu(A) - \exp[\mathbb{C}(\{\rho\}, \{A(h)\}, \phi)] \tag{3.12}$$

Theorem 3.3:

(i) $\langle \cos[\phi(m) + A(l)] \rangle_{\{\rho\}} \geqslant 0$

(ii) $\langle \cos[\phi(m) + A(l)]; \cos[\phi(n) + A(h)] \rangle_{\{\rho\}} \geqslant 0$

(iii) $\langle e^{\phi(m) + A(l)}; \cos[\phi(n) + A(h)] \rangle_{\{\rho\}} \leqslant 0$

Remarks. 1. The process $\chi \equiv (\phi, A)$ is a multicomponent Gaussian process. Theorem 3.1 applies to multicomponent processes; see I. Thus Theorem 3.3 follows from Theorem 3.1. Incidentally, the proofs are simple variants of the proof of Theorem 3.2.

2. Identity (3.8) yields obvious generalizations of our inequalities. Moreover, in Theorems 3.1(ii) and 3.3(ii) one may replace $\cos[\phi(m) + A(l)]$ by $e^{i[\phi(m)+A(l)]}$; see I.

4. QUANTUM STATISTICAL MECHANICS AND FUNCTIONAL INTEGRALS

4.1. The Uses of Gaussian and Wiener Measures

First we recall the functional integral formalism developed in detail in Refs. 12 and 7 and I. We consider N-particle systems with Hamiltonian

$$\mathrm{H}_\Lambda^{(N)} = -\sum_{i=1}^{N} (1/2m_i)\Delta_i^\Lambda + U((x)_N) \tag{4.1}$$

$$U((x)_N) = \sum_{1 \leqslant i < j \leqslant n} q_i q_j V(x_i, x_j) \tag{4.2}$$

and V is a positive (semi-) definite two-body potential.

In this subsection the spin (and other internal degrees of freedom) of the particles plays the role of a spectator and is suppressed in our notation.

We propose to express the integral kernel $\Psi_\Lambda{}^\beta((x)_N; (y)_N)$ of the operator $\exp[-\beta H_\Lambda^{(N)}]$ in terms of a combination of Wiener integrals which arise by using the Feynman–Kac formula and Gaussian functional integrals, which were used already in the classical case and in I. The path space of the Wiener measure can be chosen to be

$$\Omega = \mathop{\times}_{\tau \in [0,\infty)} \dot{\mathbb{R}}_\tau{}^\nu \tag{4.3}$$

where $\dot{\mathbb{R}}_\tau{}^\nu \simeq \dot{\mathbb{R}}^\nu$ is the one-point compactification of $\mathbb{R}^\nu$. Hence, Ω is a compact Hausdorff space, and the Borel sets generate a natural σ-algebra on Ω. The Wiener measure $P_m{}^\beta(x, y; d\omega)$, conditioned on those paths $\omega \in \Omega$ with $\omega(0) = x$, $\omega(\tau = \beta) = y$, and depending only on $\{\omega(\tau) : 0 \leqslant \tau \leqslant \beta\}$, is a σ-additive, finite measure on Ω. It is the path space measure of the process with transition function $\exp(t\Delta/2m)$. The kernel of $\exp(t\Delta/2m)$ is denoted by $p_m{}^t(x, y)$. We have

$$p_m{}^\beta(x, y) = \int_\Omega P_m{}^\beta(x, y; d\omega) \tag{4.4}$$

Let $\chi_\Lambda{}^\beta(\omega)$ be the characteristic function of the subset

$$\{\omega : \omega(\tau) \in \Lambda, \text{ for all } \tau \in [0, \beta]\} \subset \Omega$$

We set

$$P^\beta_{m,\Lambda}(x, y; d\omega) = \chi_\Lambda{}^\beta(\omega) P_m{}^\beta(x, y; d\omega) \tag{4.5}$$

This is the path space measure of the process with transition function $\exp(t\Delta^\Lambda/2m)$, where Δ^Λ is the Laplacian with 0-Dirichlet data at $\partial\Lambda$. Let

$$P_\Lambda{}^\beta((x)_N, (y)_N; d(\omega)_N) \equiv \prod_{j=1}^{N} P^\beta_{m_j,\Lambda}(x_j, y_j; d\omega_j) \tag{4.6}$$

By the Feynman–Kac formula (see, e.g., Refs. 19 and 12)

$$\psi_\Lambda{}^\beta((x)_N; (y)_N) = \int_{\Omega^{\times N}} P_\Lambda{}^\beta((x)_N, (y)_N; d(\omega)_N) \times \exp\left[-\int_0^\beta d\tau\, U((\omega(\tau))_N)\right] \tag{4.7}$$

As in I, we now express $\exp[-\int_0^\beta d\tau\, U((\omega(\tau))_N)]$ by means of a Gaussian functional integral. Let

$$\mathbb{V}(x, \tau; x', \tau') \equiv V(x, x')\, \delta(\tau - \tau') \tag{4.8}$$

Since the two-body potential V has been assumed to be positive (semi-) definite, so is $\mathbb{V}$. Let

$$L^2(\mathbb{R}_\beta^{\nu+1}) = L^2(\mathbb{R}^\nu \times [0, \beta], d^\nu x\, d\tau)$$

Let ϕ be the Gaussian process with mean 0 and covariance $\mathbb{V}$ indexed by $L^2(\mathbb{R}_\beta^{\nu+1})$. The corresponding Gaussian measure and expectation are denoted by $d\mu_{\mathbb{V}}$ and $\langle - \rangle_{\mathbb{V}}$, respectively. We recall some well-known formulas:

$$\langle \exp[i\phi(f)] \rangle_{\mathbb{V}} = \exp[-(1/2)\langle f, \mathbb{V} f\rangle]$$

$$:\exp[i\phi(f)]: \;\equiv \langle \exp[-i\phi(f)] \rangle_{\mathbb{V}}^{-1} \exp[i\phi(f)] \qquad \text{(Wick ordering)} \tag{4.9}$$

From these we obtain

$$\left\langle \prod_{j=1}^{N} :\exp[i\phi(f_j)]: \right\rangle_{\mathbb{V}} = \exp\left[-\sum_{1 \leqslant i < j \leqslant N} \langle f_i, \mathbb{V} f_j\rangle\right] \tag{4.10}$$

We assume temporarily that $V(x, y)$ is continuous in x and y and choose

$$f_j((x, \tau)) = q_j\, \delta(x - \omega_j(\tau)), \qquad j = 1, \ldots, N$$

This yields

$$\left\langle \prod_{j=0}^{N} :\exp\left[iq_j \int_0^\beta \phi(\omega_j(\tau), \tau)\, d\tau\right]: \right\rangle_{\mathbb{V}} = \exp\left\{-\int_0^\beta d\tau\, U([\omega(\tau)]_N)\right\} \tag{4.11}$$

We set

$$\alpha^{\beta}_{m,\Lambda,\phi}(x, y) \equiv \int_{\Omega} P^{\beta}_{m,\Lambda}(x, y; d\omega) :\exp\left[i \int_0^{\beta} \phi(\omega(\tau), \tau)\, d\tau \right]: \tag{4.12}$$

It should be pointed out that $\alpha^{\beta}_{m,\Lambda,\phi}(x, y)$ is really the integral kernel of the one-particle operator

$$T(\exp\{-\beta[-(1/2m)\Delta^{\Lambda} - i\phi_{\tau} - W]\})$$

where T denotes time-ordering, $i\phi_{\tau}(\cdot) \equiv i\phi(\cdot, \tau)$ is a purely imaginary one-particle potential, and $W(x) = \frac{1}{2}V(x, x)$.

From (4.7), (4.11), and (4.12) we deduce

$$\psi_{\Lambda}{}^{\beta}((x)_N; (y)_N) = \left\langle \prod_{j=1}^{N} \alpha^{\beta}_{m_j,\Lambda,q_j\phi}(x_j, y_j) \right\rangle_{V} \tag{4.13}$$

Inserting this into (1.6) and (1.8), one obtains an expression for the partition function and the RDMs in terms of Wiener and Gaussian integrals.

4.2. Taking into Account Statistics: An Exercise in Multilinear Algebra

The purpose of this subsection is to express the partition function and the RDMs (or ITGFs) of systems with Bose or Fermi statistics in compact form in terms of Gaussian integrals ("Boltzmann statistics" has been treated in I). This will permit us to apply the correlation inequalities of Section 3 to construct and investigate the thermodynamic limit (at least for Bose gases). We start with stating the main results of this subsection. We consider a system of finitely many species of particles with Bose ($\epsilon = +1$) or Fermi ($\epsilon = -1$) statistics. First it is assumed that the particles are spinless, but at the close of this subsection we show how one can incorporate spin. The Hamiltonian $H^{(\cdot)}_{\Lambda}$ is as in (4.1) with $m_1 = \cdots = m_{i_1}, \ldots, m_{i_{l-1}+1} = \cdots = m_N$, $q_1 = \cdots = q_{i_1}, \ldots, q_{i_{l-1}+1} = \cdots = q_N$, and l is the number of species.

We define

$$\rho_{\Lambda,\epsilon}(m, z; x, y; \phi) = \sum_{j=1}^{\infty} \frac{\epsilon^{j-1}}{j} z^j \int_{\Omega} P^{j\beta}_{m,\Lambda}(x, y; d\omega) \times \prod_{k=0}^{j-1} :\exp\left[i \int_0^{\beta} \phi(\omega(\tau + k\beta), \tau)\, d\tau \right]: \tag{4.14}$$

$$S_{\Lambda,\epsilon}(m, z; \phi) = \int_{\Lambda} d^{\nu}x\, \rho_{\Lambda,\epsilon}(m, z; x, x; \phi) \tag{4.15}$$

We assume (at least temporarily) that

$$K \equiv \sup_{x \in \mathbb{R}^\nu} V(x, x) < \infty \tag{4.16}$$

Since by (4.9)

$$\left| :\exp\left[\pm iq \int_0^\beta \phi(\omega(\tau), \tau)\, d\tau \right]: \right| \leqslant \exp\left[(q^2/2) \int_0^\beta V(\omega(\tau), \omega(\tau))\, d\tau \right] \leqslant \exp(\beta q^2 K/2) \tag{4.17}$$

the series on the rhs of (4.14) converges absolutely if

$$z e^{\beta K/2} < 1 \tag{4.18}$$

For Fermi statistics one can relax conditions (4.16) and (4.18). See Sections 2 and 5. But for the time being they are imposed without further mention. Let $A = (A_{ij})$ be some $N \times N$ matrix. We define

$$\delta_\epsilon^{(N)}(A_{ij}) = \sum_{\pi \in S_N} \epsilon^{\sigma(\pi)} \prod_{j=1}^N A_{j\pi(j)} \tag{4.19}$$

where $\sigma(\pi)$ is the signature of the permutation π. Clearly

$$\delta_{-1}^{(N)}(A_{ij}) = \det(A), \qquad \delta_{+1}^{(N)}(A_{ij}) = \operatorname{perm}(A) \tag{4.20}$$

Theorem 4.1. Consider a system of l species of (spinless) particles with statistics ϵ_k, mass m_k, charge q_k, and activity z_k, $k = 1,..., l$, in the grand canonical ensemble at inverse temperature β. Let $\boldsymbol{\epsilon} = (\epsilon_1,..., \epsilon_l)$, $\mathbf{z} = (z_1,..., z_l)$. Then the partition function is given by

$$\Xi_{\Lambda,\boldsymbol{\epsilon}}(\beta, \mathbf{z}) = \left\langle \exp\left[\sum_{k=1}^l S_{\Lambda,\epsilon_k}(m_k, z_k; q_k\phi) \right] \right\rangle_{\mathbb{V}} \tag{4.21}$$

and the correlation functions by

$$\rho_{\Lambda,\boldsymbol{\epsilon}}(\beta, \mathbf{z}; (x^1)_{i_1} \cdots (x^l)_{i_l}, (y^1)_{i_1} \cdots (y^l)_{i_l}) = \left\langle \prod_{k=1}^l \delta_{\epsilon_k}^{(i_k)}\left(z \frac{\partial}{\partial z} \rho_{\Lambda,\epsilon_k}(\beta, z_k; x_i^k, y_j^k; q_k\phi) \right) \right\rangle_{\Lambda,\boldsymbol{\epsilon}} (\beta, \mathbf{z}) \tag{4.22}$$

where

$$\langle - \rangle_{\Lambda,\boldsymbol{\epsilon}}(\beta, \mathbf{z}) = \Xi_{\Lambda,\boldsymbol{\epsilon}}(\beta, \mathbf{z})^{-1} \left\langle - \exp\left[\sum_{k=1}^l S_{\Lambda,\epsilon_k}(m_k, z_k; q_k\phi) \right] \right\rangle_{\mathbb{V}} \quad \blacksquare$$

For later purposes we explicitly consider the special case of charge-conjugation-invariant systems of two species of particles; see Definition 1.1. We define

$$\mathbb{C}_{\Lambda,\epsilon}(\beta, z; q\phi) = S_{\Lambda,\epsilon}(m, z; q\phi) + S_{\Lambda,\epsilon}(m, z; -q\phi) \tag{4.23}$$

Theorem 4.1 then takes the following form:

Theorem 4.1′. For the charge-conjugation-invariant systems introduced in (1.4)–(1.8)

$$\Xi_{\Lambda,\epsilon}(\beta, z) = \langle \exp \mathbb{C}_{\Lambda,\epsilon}(\beta, z; q\phi)\rangle_{\mathbb{V}} \tag{4.21'}$$

$$\begin{aligned} &\rho_{\Lambda,\epsilon}(\beta, z; (x)_N(x')_M; (y)_N(y')_M) \\ &= \Big\langle \delta_\epsilon^{(N)}\Big(z\frac{\partial}{\partial z}\rho_{\Lambda,\epsilon}(\beta, z; x_i, y_j; q\phi)\Big) \\ &\quad\times \delta_\epsilon^{(M)}\Big(z\frac{\partial}{\partial z}\rho_{\Lambda,\epsilon}(\beta, z; x_i', y_j'; -q\phi)\Big)\Big\rangle_{\Lambda,\epsilon}(\beta, z) \end{aligned} \tag{4.22'}$$

where

$$\langle - \rangle_{\Lambda,\epsilon}(\beta, z) = \Xi_{\Lambda,\epsilon}(\beta, z)^{-1}\langle - \exp \mathbb{C}_{\Lambda,\epsilon}(\beta, z; q\phi)\rangle_{\mathbb{V}}$$

Remarks. 1. The expectation $\langle - \rangle_{\Lambda,\epsilon}(\beta, z)$ defined in (4.22′) is given by a positive probability measure, because the "action" $\mathbb{C}_{\Lambda,\epsilon}(\beta, z; q\phi)$ is real-valued. See (4.14), (4.15), (4.23).

2. If the system is not charge conjugation invariant, as in Theorem 4.1, then $\langle - \rangle_{\Lambda,\epsilon}(\beta, \mathbf{z})$ is given by a complex measure.

3. Expressions for ITGFs similar to the ones given in Theorem 4.1 for the RDMs can be derived, too, but are more complicated; see Section 4.3.

4. Spin is incorporated at the end of this section.

Proof of Theorem 4.1. The opening move in this proof consists of first reformulating Theorem 4.1 in a more reasonable terminology. It then follows from standard identities of multilinear algebra, which, for the convenience of the reader, we briefly review in Section 4.3.

First, we notice that it really suffices to prove Theorem 4.1 for one species of particles only. The case of many species will turn out to be an obvious generalization.

We define

$$A_{q\phi} \equiv A^\beta_{m,q\phi} = T\exp[-\beta(-1/2m)\Delta^\Lambda - iq\phi_\tau - q^2 W)] \tag{4.24}$$

where $W(x) = \frac{1}{2}V(x, x)$. By the Feynman–Kac formula, the integral kernel of $A_{q\phi}$ is given by

$$\begin{aligned} A_{q\phi}(x, y) &\equiv \alpha^\beta_{m,\Lambda,q\phi}(x, y) \\ &= \int_\Omega P^\beta_{m,\Lambda}(x, y; d\omega) :\exp\Big[i\int_0^\beta \phi(\omega(\tau), \tau)\, d\tau\Big]: \end{aligned} \tag{4.25}$$

[see formula (4.12)].

In order to express the kernel of $(A_{q\phi})^j$ we use the following well-known result:

Lemma 4.2.

$$\int \prod_{j=1}^{n} d^\nu x_j \int_{\Omega^{\times(n+1)}} \prod_{j=1}^{n+1} \left\{ P_{m,\Lambda}^{\beta}(x_{j-1}, x_j; d\omega_j) :\exp\left[iq \int_0^\beta \phi(\omega_j(\tau), \tau)\, d\tau \right]: \right\}$$

$$= \int_\Omega P_\Lambda^{(n+1)\beta}(x_0, x_{n+1}; d\omega) \prod_{j=0}^{n} :\exp\left[iq \int_0^\beta \phi(\omega(\tau + j\beta), \tau)\, d\tau \right]:$$

Proof. An immediate consequence of the semigroup property of $\exp[t(1/2m)\Delta^\Lambda]$ and the Feynman–Kac formula. ■

Thus

$$(A_{q\phi})^j(x, y) = \int_\Omega P^{j\beta}(x, y; d\omega) \prod_{k=0}^{j-1} :\exp\left[iq \int_0^\beta \phi(\omega(\tau + k\beta), \tau)\, d\tau \right]: \tag{4.26}$$

This identity and (4.14) yield

$$\rho_{\Lambda,\epsilon}(\beta, z; x, y; \phi) = \sum_{j=1}^{\infty} \frac{\epsilon^{j-1}}{j} z^j (A_\phi)^j(x, y) = -\epsilon \ln(1 - \epsilon z A_\phi)(x, y) \tag{4.27}$$

and

$$z \frac{\partial}{\partial z} \rho_{\Lambda,\epsilon}(\beta, z; x, y; \phi) = z\{(1 - \epsilon z A_\phi)^{-1} A_\phi\}(x, y) \tag{4.28}$$

Furthermore, by Lemma 4.2, (4.14), and (4.15),

$$\begin{aligned} \exp S_{\Lambda,\epsilon}(m, z; \phi) &= \exp \int_\Lambda d^\nu x\, \rho_{\Lambda,\epsilon}(m, z; x, x; \phi) \\ &= \exp\left[-\epsilon \int_\Lambda d^\nu x \ln(1 - \epsilon z A_\phi)(x, x) \right] \\ &= \exp[-\epsilon \operatorname{Tr} \ln(1 - \epsilon z A_\phi)] = \det(1 - \epsilon z A_\phi)^{-\epsilon} \end{aligned} \tag{4.29}$$

In this reformulation, Theorem 4.1 maintains

$$\Xi_{\Lambda,\epsilon}(\beta, z) = \langle \det(1 - \epsilon z A_{q\phi})^{-\epsilon} \rangle_{\mathbb{V}} \tag{4.30}$$

by (4.21) and (4.29), and

$$\rho_{\Lambda,\epsilon}(\beta, z; (x)_N, (y)_N) = \langle \delta_\epsilon^{(N)}(z\{(1 - \epsilon z A_{q\phi})^{-1} A_{q\phi}\}(x_i, y_j)) \rangle_{\Lambda,\epsilon}(\beta, z) \tag{4.31}$$

where $\delta_\epsilon^{(N)}$ is defined in (4.19), and $\langle - \rangle_{\Lambda,\epsilon}(\beta, z)$ in (4.22); see (4.28). Next, using (4.12), (4.13), and (4.25), we see that

$$\psi_\Lambda{}^\beta((x)_N; (y)_N) = \left\langle \prod_{j=1}^{N} A_{q\phi}(x_j, y_j) \right\rangle_{\mathbb{V}} \tag{4.32}$$

In formula (1.6) we expressed $\Xi_\Lambda(\beta, z)$ by

$$\Xi_{\Lambda,\epsilon}(\beta, z) = \sum_{N=0}^{\infty} \frac{z^N}{N!} \sum_{\pi \in S_N} \epsilon^{\sigma(\pi)} \int d(x)_N \, \psi_\Lambda{}^\beta((x)_N, {}^\pi(x)_N)$$

Hence, by (4.32) and definition (4.19) of $\delta_\epsilon^{(N)}$,

$$\Xi_{\Lambda,\epsilon}(\beta, z) = \left\langle \sum_{N=0}^{\infty} \frac{1}{N!} \int d(x)_N \, \delta_\epsilon^{(N)}(zA_{q\phi}(x_i, x_j)) \right\rangle_{\mathbb{V}} \tag{4.33}$$

and we have interchanged taking $\langle - \rangle_{\mathbb{V}}$ and $\sum_{N=0}^{\infty} (1/N!) \int d(x)_N -$ [this is permitted if $z \exp(\beta q^2 K/2) < 1$; see (4.18)]. The equality of the right sides of (4.33) and (4.30) is well known (in the sense of formal power series it holds in general, and if $z \exp(\beta q^2 K/2) < 1$ both right sides are well defined). See also Section 4.3. Next, by formulas (1.8), (4.32), and (4.19),

$$\begin{aligned}
&\rho_{\Lambda,\epsilon}(\beta, z; (x)_N, (y)_N) \\
&\quad = \Xi_{\Lambda,\epsilon}(\beta, z)^{-1} \sum_{N'=0}^{\infty} \frac{z^{N+N'}}{N'!} \int d(u)_{N'} \sum_{\pi \in S_{N+N'}} \epsilon^{\sigma(\pi)} \\
&\qquad \times \Psi_\Lambda{}^\beta((x)_N(u)_{N'}; \pi((y)_N(u)_{N'})) \\
&\quad = \Xi_{\Lambda,\epsilon}(\beta, z)^{-1} \sum_{N'=0}^{\infty} \frac{1}{N'!} \left\langle \int d(u)_{N'} \, \delta_\epsilon^{(N+N')}(zA_{q\phi}(v_i, w_j)) \right\rangle_{\mathbb{V}}
\end{aligned} \tag{4.34}$$

where

$$(v_1, \ldots, v_N, v_{N+1}, \ldots, v_{N+N'}) = (x_1, \ldots, x_N, u_1, \ldots, u_{N'})$$

$$(w_1, \ldots, w_N, w_{N+1}, \ldots, w_{N+N'}) = (y_1, \ldots, y_N, u_1, \ldots, u_{N'})$$

The reader familiar with multilinear algebra will recognize the rhs of (4.34) as being identical to the rhs of (4.31). If we finally insert (4.28) into the rhs of (4.31), the proof of Theorem 4.1 is complete for the case of one species of particles. The case of finitely many species follows in the obvious way. ■

We conclude this subsection by showing how to incorporate spin in this formalism. Again, it clearly suffices to consider the special case of one species of particle. The Hilbert space of the spin degree of freedom of one particle is $\mathbb{C}^{2S+1}$, with S the total spin. We choose an orthonormal basis $\{\phi_\mu\}_{\mu=-S}^{S}$ in $\mathbb{C}^{2S+1}$ labeled by the eigenvalues μ of one component of the spin operator. The basic fact to be noticed is that the total Hamiltonian $H_\Lambda^{(N)}$ [see (4.1)] is spin-independent (although that is not absolutely crucial for the existence of a functional integral formalism, as mentioned in Remark 2, Section 1.3). Let $r = (x, \mu)$, and define

$$A^S_{q\phi} = A_{q\phi} \otimes \mathbb{1}_S \tag{4.24'}$$

where $A_{q\phi}$ is given by (4.24) and (4.25), and $\mathbb{1}_S$ is the unit matrix on $\mathbb{C}^{2S+1}$. The integral kernel of $A^S_{q\phi}$ is given by

$$A^S_{q\phi}(r, r') \equiv A^S_{q\phi}((x, \mu), (x', \mu')) = A_{q\phi}(x, x')\, \delta_{\mu,\mu'}$$

Since

$$\mathrm{Tr}_{\mathbb{C}^{2S+1}}(\mathbb{1}_S) = 2S + 1$$

we have

$$\mathrm{Tr}\ln(1 - \epsilon z A^S_{q\phi}) = (2S + 1)\,\mathrm{Tr}\ln(1 - \epsilon z A_{q\phi})$$

so that

$$\det(1 - \epsilon z A^S_{q\phi})^{-\epsilon} = \det(1 - \epsilon z A_{q\phi})^{-\epsilon(2S+1)}$$

The proof of Theorem 4.1 extends to the case of a system of particles with spin and, together with the above remarks, gives (see also Section 4.3) the following result:

Theorem 4.1″.

$$\Xi_\Lambda(\beta, z) = \langle \det(1 - \epsilon z A^S_{q\phi})^{-\epsilon}\rangle_{\mathbb{V}} = \langle \det(1 - \epsilon z A_{q\phi})^{-\epsilon(2S+1)}\rangle_{\mathbb{V}} \tag{4.21″}$$

$$\rho_{\Lambda,\epsilon}(\beta, z; (r)_N(r')_N) = \langle \delta^{(N)}_\epsilon(z\{(1 - \epsilon z A^S_{q\phi})^{-1} A^S_{q\phi}\}(r_i, r_j'))\rangle_{\Lambda,\epsilon}(\beta, z) \tag{4.22″}$$

where

$$\langle - \rangle_{\Lambda,\epsilon}(\beta, z) = \Xi_\Lambda(\beta, z)^{-1}\langle - \det(1 - \epsilon z A_{q\phi})^{-\epsilon(2S+1)}\rangle_{\mathbb{V}} \quad \blacksquare$$

The purpose of the next subsection is to briefly review some multilinear algebra, sketch the proof of the above identities, and find compact expressions for correlation functions and ITGFs.

4.3. Exercises in Multilinear Algebra

Let $\mathscr{H}$ be a complex Hilbert space of dimension $n \leqslant \infty$, $\{u_i\}_{i=0}^n$ a complete orthonormal system in $\mathscr{H}$, and $f_1,\ldots, f_N$ and $g_1,\ldots, g_N$ vectors in $\mathscr{H}$. Let A be a trace-class operator on $\mathscr{H}$, i.e., $\|A\|_1 = \mathrm{Tr}|A| < \infty$. The symbol $\otimes_\epsilon$ denotes the symmetric tensor product if $\epsilon = +1$ and the antisymmetric tensor product if $\epsilon = -1$. We define

$$\mathscr{H}^{\otimes_\epsilon m} = \underbrace{\mathscr{H} \otimes_\epsilon \cdots \otimes_\epsilon \mathscr{H}}_{m \text{ times}}, \qquad A^{\otimes_\epsilon m} = \underbrace{A \otimes_\epsilon \cdots \otimes_\epsilon A}_{m \text{ times}}$$

$$\mathscr{H}^{\otimes_\epsilon 0} \equiv \mathbb{C}, \qquad A^{\otimes_\epsilon 0} \equiv 1$$

For $f_1,\ldots,f_m$ in $\mathscr{H}$ we set

$$\prod_{j=1}^{m}{}^{\otimes_\epsilon} f_j = f_1 \otimes_\epsilon \cdots \otimes_\epsilon f_m$$

The scalar product on $\mathscr{H}^{\otimes_\epsilon m}$ is denoted $\langle \cdot, \cdot \rangle$, for all $m = 0, 1, 2,\ldots$.

We introduce an "unnormalized, reduced density matrix" ρ_ϵ by

$$\begin{aligned}\rho_\epsilon(\bar{g}_1,\ldots,\bar{g}_N,f_1,\ldots,f_N) &\equiv \sum_{m=0}^{\infty} \frac{(m+N)!}{m!} \sum_{i_1\cdots i_m} \Big\langle \prod_{j=1}^{m}{}^{\otimes_\epsilon} u_{i_j} \otimes_\epsilon \prod_{k=1}^{N}{}^{\otimes_\epsilon} g_k, \\ &\qquad A^{\otimes_\epsilon(m+N)} \prod_{j=1}^{m}{}^{\otimes_\epsilon} u_{i_j} \otimes_\epsilon \prod_{k=1}^{N} f_k \Big\rangle \\ &= \sum_{m=0}^{\infty} \frac{(m+N)!}{m!} \sum_{i_1\cdots i_m} \Big\langle \prod_{j=1}^{m}{}^{\otimes_\epsilon} u_{i_j} \otimes_\epsilon \prod_{k=1}^{N}{}^{\otimes_\epsilon} g_k, \\ &\qquad \prod_{j=1}^{m}{}^{\otimes_\epsilon} (Au_{i_j}) \otimes_\epsilon \prod_{k=1}^{N}{}^{\otimes_\epsilon} (Af_k) \Big\rangle \end{aligned} \tag{4.35}$$

When $\epsilon = +1$ (Bose statistics) we assume that $\|A\| < 1$.

Lemma 4.3:

$$\rho_\epsilon(\bar{g}_1,\ldots,\bar{g}_N,f_1,\ldots,f_N) = \det(1-\epsilon A)^{-\epsilon}\, \delta_\epsilon^{(N)}(\langle g_i, (1-\epsilon A)^{-1} A f_j\rangle)$$

Proof. It suffices to prove Lemma 4.3 in the finite-dimensional case. The proof for the infinite-dimensional case follows by a standard limiting argument, provided $\|A\|_1 < \infty$, and $\|A\| < 1$, when $\epsilon = +1$.

We first consider Bose statistics, $\epsilon = +1$. Let $\xi = \xi^1 + i\xi^2$ be the complex Gaussian process with mean 0 and covariance 1, indexed by $\mathscr{H}$; i.e.,

$$\int d\mu(\xi)\, \xi(f) = 0, \qquad \int d\mu(\xi)\, \xi(\bar{f})\bar{\xi}(g) = 2\langle f, g\rangle \tag{4.36}$$

where

$$d\mu(\xi) = (2\pi)^{-n} \exp[-\tfrac{1}{2}\langle \xi, \xi\rangle] \prod_{\alpha=1}^{n} d\xi_\alpha{}^1\, d\xi_\alpha{}^2$$

is the normalized Gaussian measure.

By (4.35) and (4.36)

$$\begin{aligned}\rho_1(\bar{g}_1,\ldots,\bar{g}_N,f_1,\ldots,f_N) &= \sum_{m=0}^{\infty} \frac{1}{2^m m!} \sum_{i_1\cdots i_m} \int \prod_{k=1}^{N} \tfrac{1}{2}\xi(\bar{g}_k)\bar{\xi}(Af_k) \prod_{j=1}^{m} \xi(\bar{u}_{i_j})\bar{\xi}(Au_{i_j})\, d\mu(\xi)\end{aligned}$$

Note that $\sum_{j=1}^{n} \xi(\bar{u}_j)\bar{\xi}(Au_j) = \langle \xi, A\xi \rangle$. Thus

$$\begin{aligned}
&\rho_1(\bar{g}_1,..., \bar{g}_N, f_1,..., f_N) \\
&\quad = \sum_{m=0}^{\infty} \frac{1}{2^m m!} \int \langle \xi, A\xi \rangle^m \prod_{k=1}^{N} \tfrac{1}{2}\xi(\bar{g}_k)\bar{\xi}(Af_k)\, d\mu(\xi) \\
&\quad = (2\pi)^{-n} \int \prod_{k=1}^{N} \tfrac{1}{2}\xi(\bar{g}_k)\bar{\xi}(Af_k) \exp[-\tfrac{1}{2}\langle \xi, (1-A)\xi \rangle] \prod_{\alpha=1}^{n} d\xi_\alpha{}^1\, d\xi_\alpha{}^2 \\
&\quad = (2\pi)^{-n} \int \exp[-\tfrac{1}{2}\langle \xi, (1-A)\xi \rangle] \prod_{\alpha=1}^{n} d\xi_\alpha{}^1\, d\xi_\alpha{}^2 \\
&\qquad \times \delta_1^{(N)}(\langle g_i, (1-A)^{-1}Af_j \rangle) \\
&\quad = \det(1-A)^{-1}\, \delta_1^{(N)}(\langle g_i, (1-A)^{-1}Af_j \rangle)
\end{aligned}$$

This completes the proof for $\epsilon = 1$.

Next, we consider Fermi statistics, $\epsilon = -1$. Let $\psi_\alpha{}^1$ and $\psi_\alpha{}^2$, $\alpha = 1,..., n$, be totally anticommuting variables, and let $\int$ be the Berezin integral, which may be defined by the property that $\int \exp\langle \psi^1, A\psi^2 \rangle = \det(A)$, where

$$\langle \psi^1, A\psi^2 \rangle = \sum_{\alpha,\gamma=1}^{n} \psi_\alpha{}^1 A_{\alpha\gamma} \psi_\gamma{}^2$$

(see, e.g., Ref. 28). It is known and follows easily from the above definition of the Berezin integral by differentiation that

$$\begin{aligned}
\rho_{-1}(\bar{g}_1,..., \bar{g}_N, f_1,..., f_N) &= \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{i_1 \cdots i_m} \int \prod_{k=1}^{N} \psi^1(\bar{g}_k)\psi^2(Af_k) \\
&\qquad \times \prod_{j=1}^{m} \psi^1(\bar{u}_{i_j})\psi^2(Au_{i_j}) \exp\langle \psi^1, \psi^2 \rangle \\
&= \int \prod_{k=1}^{N} \psi^1(\bar{g}_k)\psi^2(Af_k) \exp\langle \psi^1, (1+A)\psi^2 \rangle \\
&= \delta_{-1}^{(N)}(\langle g_i, (1+A)^{-1}Af_j \rangle) \det(1+A) \quad \blacksquare
\end{aligned}$$

Remark. The purpose of introducing the Gaussian (resp. Berezin) integral is merely to reduce somewhat lengthy combinatorics to known properties of those integrals. It could be avoided completely.

Next, we introduce the standard Fock space

$$\mathscr{F}_\epsilon = \bigoplus_{m=0}^{\infty} \mathscr{H}^{\otimes_\epsilon m}$$

and define the operator $\Gamma_\epsilon(A)$ on $\mathscr{F}_\epsilon$ by

$$\Gamma_\epsilon(A) = \bigoplus_{m=0}^{\infty} A^{\otimes_\epsilon m} \tag{4.37}$$

Note that

$$\Gamma_\epsilon(A)\Gamma_\epsilon(B) = \Gamma_\epsilon(A \cdot B) \tag{4.38}$$

We define

$$d\Gamma_\epsilon(A) = \frac{d}{dt}\Gamma_\epsilon(e^{tA})|_{t=0} \tag{4.39}$$

This is Segal's formulation of "second quantization"; see, e.g., Ref. 29. As a corollary of Lemma 4.3, we have

$$\mathrm{Tr}[\Gamma_\epsilon(A)] = \det(1 - \epsilon A)^{-\epsilon} \tag{4.40}$$

[Set $N = 0$ in Lemma 4.3 and use (4.35) and (4.37). A direct proof of (4.40) not involving Gaussian (resp. Berezin) integrals is easily found: By analyticity, it suffices to prove (4.40) for self-adjoint A. Both sides in (4.40) are unitary invariants. Thus one may choose A to be diagonal. Then (4.40) becomes a trivial exercise.]

We may now define "correlation functions" and "ITGFs." The former are given by

$$\left\langle \prod_{j=1}^{m} d\Gamma_\epsilon(B_j) \right\rangle_A \equiv [\mathrm{Tr}\,\Gamma_\epsilon(A)]^{-1}\,\mathrm{Tr}\left[\Gamma_\epsilon(A) \prod_{j=1}^{m} d\Gamma(B_j)\right] \tag{4.41}$$

The latter are given by

$$\left\langle \prod_{j=1}^{m} d\Gamma_\epsilon(B_j)_{\tau_j} \right\rangle_A \equiv [\mathrm{Tr}\,\Gamma_\epsilon(A)]^{-1} \times \mathrm{Tr}\left[\Gamma_\epsilon(A)^{1-\tau_m+\tau_1} \prod_{j=1}^{m} \{d\Gamma(B_j)\,\Gamma_\epsilon(A)^{\tau_{j+1}-\tau_j}\}\right] \tag{4.42}$$

where $0 \leqslant \tau_1 \leqslant \tau_2 \leqslant \cdots \leqslant \tau_m \leqslant 1$, and $A = e^h$, for some operator h with $\mathrm{Re}\, h < 0$. Obviously

$$\left\langle \prod_{j=1}^{m} d\Gamma_\epsilon(B_j) \right\rangle_A = \left\langle \prod_{j=1}^{m} d\Gamma_\epsilon(B_j)_{\tau_j=0} \right\rangle_A$$

so that it suffices to calculate the rhs of (4.42). By (4.39) and (4.38),

$$\left\langle \prod_{j=1}^{m} d\Gamma_\epsilon(B_j)_{\tau_j} \right\rangle_A = \mathrm{Tr}[\Gamma_\epsilon(A)]^{-1} \frac{\partial^m}{\partial s_1 \cdots \partial s_m} \times \mathrm{Tr}\left[\Gamma_\epsilon\left(A^{1-\tau_m+\tau_1} \prod_{j=1}^{m} e^{s_j B_j} A^{\tau_{j+1}-\tau_j}\right)\right]\Bigg|_{s_1=\cdots=s_m=0} \tag{4.43}$$

Next note that

$$\operatorname{Tr}\Gamma_\epsilon(A) = \det(1 - \epsilon A)^{-\epsilon} = \exp[-\epsilon \operatorname{Tr}\ln(1 - \epsilon A)]$$

Thus, using Leibniz' rule,

$$\left\langle \prod_{j=1}^{m} d\Gamma_\epsilon(B_j)_{\tau_j} \right\rangle_A = \sum_{\substack{\text{partitions} \\ C_1,\ldots,C_M}} \prod_{j=1}^{M} \left[-\epsilon \prod_{k \in C_j} \frac{\partial}{\partial s_k} \ln\left(1 - \epsilon A^{1-\tau_m+\tau_1} \times \prod_{j=1}^{m} (e^{s_j B_j} A^{\tau_{j+1}-\tau_j})\right)\right]\Bigg|_{s_1=\cdots=s_m=0} \tag{4.44}$$

The rhs can be calculated by using the formulas

$$\frac{d}{ds}\operatorname{Tr}\ln[1 - \epsilon A(s)] = -\epsilon \operatorname{Tr}\{[1 - \epsilon A(s)]^{-1}A'(s)\}$$

$$\frac{d}{ds}[1 - \epsilon A(s)]^{-1} = \epsilon[1 - \epsilon A(s)]^{-1}A'(s)[1 - \epsilon A(s)]^{-1}$$

with $A'(s) = (d/ds)A(s)$.

Finally, we note a connection between correlation functions and reduced density matrices: Let B_j, $j = 1,\ldots, N$, be given by

$$B_j g = \langle g_j, g\rangle f_j \qquad \text{for all } g \in \mathscr{H}$$

Let ρ_ϵ be given by (4.35) and $\langle - \rangle_A$ by (4.41). Then

$$\rho_\epsilon(\bar{g}_1,\ldots, \bar{g}_N, f_1,\ldots, f_N) = \operatorname{Tr}[\Gamma_\epsilon(A)]\left\langle : \prod_{j=1}^{N} d\Gamma_\epsilon(B_j) : \right\rangle_A \tag{4.45}$$

where :–: is the usual Wick order of products of operators on $\mathscr{F}_\epsilon$. [The rhs of (4.45) can be calculated by using (4.41) and (4.44). Comparison with Lemma 4.2 then completes the proof of (4.45).]

5. THE THERMODYNAMIC LIMIT: UNIFORM BOUNDS, EXISTENCE, AND PROPERTIES

Stability in the grand canonical ensemble and existence of the thermodynamic limit of the pressure have been discussed in Section 2. The main purpose of this section is to derive upper and lower bounds on correlation functions which are uniform in Λ, prove the existence of the thermodynamic limit of the correlation functions of charge-conjugation-invariant Bose systems (provided the activity is so small that the system is stable), and estimate correlation functions of systems with Fermi statistics by those of Bose systems (resp. correlation functions of Bose systems in a magnetic field by those of systems in zero magnetic field).

Unless mentioned otherwise, the systems are assumed to be charge conjugation invariant. The particles may have spin.

5.1. Uniform Upper Bounds on Partition and Correlation Functions

The main auxiliary estimates required in this section have been already derived in I, Section 2.2. Thus, we may be brief. In Section 4.2, (4.17) we have shown that

$$\left| :\exp\left[iq \int_0^\beta d\tau\, \phi(\omega(\tau), \tau) \right]: \right| \leqslant \exp(\beta q^2 K/2) \tag{5.1}$$

with $K = \sup_{x \in \mathbb{R}^\nu} V(x, x)$. Furthermore, by (4.5)

$$\int_\Omega P^{j\beta}_{m,\Lambda}(x, y; d\omega) \leqslant \int_\Omega P^{j\beta}_m(x, y; d\omega) = \left(\frac{2\pi m}{j\beta}\right)^{\nu/2} \exp\left[-\frac{|x-y|^2}{2j\beta} \right] \tag{5.2}$$

Using (5.1) and (5.2), we propose to estimate:

(I) $|\rho_{\Lambda,\epsilon}(\beta, z; x, y; q\phi)| = |\ln(1 - \epsilon z A_{q\phi})(x, y)|$

[see (4.14) and (4.27)].

(II) $\left| z \dfrac{\partial}{\partial z} \rho_{\Lambda,\epsilon}(\beta, z; x, y; q\phi) \right| = |z\{(1 - \epsilon z A_{q\phi})^{-1} A_{q\phi}\}(x, y)|$

[see (4.28)].

(III) $|\exp S_{\Lambda,\epsilon}(m, z; q\phi)| = |\det(1 - \epsilon z A_{q\phi})^{-\epsilon}|$

[see (4.15) and (4.29)].

(I) By (4.26), (4.27), (5.1), and (5.2)

$$\begin{aligned} |\rho_{\Lambda,\epsilon}(\beta, z; x, y; \phi)| &\leqslant \sum_{j=1}^\infty \frac{|z|^j}{j} |(A_{q\phi})^j(x, y)| \\ &\leqslant \left(\frac{2\pi m}{\beta}\right)^{\nu/2} \sum_{j=1}^\infty \frac{[|z| \exp(\beta q^2 K/2)]^j}{j^{1+\nu/2}} \exp\left[-\frac{|x-y|^2}{2j\beta} \right] \\ &\leqslant \rho(\beta, z, q) \end{aligned} \tag{5.3}$$

where

$$\rho(\beta, z, q) \equiv \left(\frac{2\pi m}{\beta}\right)^{\nu/2} \sum_{j=1}^\infty \frac{[|z| \exp(\beta q^2 K/2)]^j}{j^{1+\nu/2}} \tag{5.4}$$

which is finite, provided

$$|z| \exp(\beta q^2 K/2) < 1 \tag{5.5}$$

(II) By (4.26), (5.1), and (5.2)

$$|z\{(1-\epsilon z A_{q\phi})^{-1}A_{q\phi}\}(x,y)| \leqslant \sum_{j=1}^{\infty} |z|^{j}|(A_{q\phi})^{j}(x,y)| \leqslant \tilde{\rho}(\beta, z, q; x, y) \quad (5.6)$$

where

$$\tilde{\rho}(\beta, z, q; x, y) \equiv \left(\frac{2\pi m}{\beta}\right)^{\nu/2} \sum_{j=1}^{\infty} \frac{[|z| \exp(\beta q^2 K/2)]^j}{j^{\nu/2}} \exp\left[-\frac{|x-y|^2}{2j\beta}\right] \quad (5.7)$$

and the rhs converges if (5.5) is satisfied.

It is trivial to check that (5.3)–(5.7) remain true if $A_{q\phi}$ is replaced by $A^S_{q\phi}$ [see Section 4.2, (4.24′)], i.e., if spin is included.

(III) We first present an upper bound that holds for $\epsilon = \pm 1$:

$$|\det(1-\epsilon z A_{q\phi})^{-\epsilon}| \leqslant \exp|\mathrm{Tr}\ln(1-\epsilon z A_{q\phi})| \leqslant \exp[\rho(\beta, z, q)|\Lambda|] \quad (5.8)$$

as follows from (4.29) and (5.3). The rhs of (5.8) is finite if $|z| \exp(\beta q^2 K/2) < 1$. Under this condition identity (4.30), i.e.,

$$\Xi_{\Lambda,\epsilon}(\beta, z) = \langle \det(1-\epsilon z A_{q\phi})^{-\epsilon}\rangle_V \quad (5.9)$$

holds rigorously as an equation between holomorphic functions of z with $|z| < \exp(-\beta q^2 K/2)$, as follows from (5.3) and Lemma 4.3 by a simple limiting argument. By analyticity in z of both sides in Eq. (5.9), this identity remains true for all $z > 0$ for which $|\det(1-\epsilon z A_{q\phi})^{-\epsilon}|$ is bounded uniformly in ϕ, at the least. From Theorem 2.2, (2.5)–(2.6), we know that for $\epsilon = 1$ (Bose statistics) the domain of holomorphy of

$$\langle \det(1-zA_{q\phi})^{-1}\det(1-zA_{-q\phi})^{-1}\rangle_V$$

does not include the whole positive, real axis. Indeed, given $\delta > 0$, there is a bounded region (e.g., a cube) Λ_δ such that $\Xi_{\Lambda,1}(\beta, z)$ is divergent at $z = 1+\delta$ for all $\Lambda \supset \Lambda_\delta$. Therefore $\det(1-zA_{q\phi})^{-1}\det(1-zA_{-q\phi})^{-1}$ and thus $|\det(1-zA_{q\phi})^{-1}|$ cannot be bounded uniformly in ϕ for $z = 1+\delta$, $\Lambda \supset \Lambda_\delta$. [However, for superstable potentials V,

$$\Xi_{\Lambda,1}(\beta, z) = \langle \det(1-zA_{q\phi})^{-1}\rangle_V$$

exists for all $z > 0$.]

Next, we set $\epsilon = -1$ (Fermi statistics). Then

$$\begin{aligned} |\det(1+zA_{q\phi})| &= \det\{(1+zA_{q\phi})(1+zA^*_{q\phi})\}^{1/2} \\ &= \exp\{\tfrac{1}{2}\mathrm{Tr}\ln[1+z(A_{q\phi}+A^*_{q\phi})+z^2|A_{q\phi}|^2]\} \\ &\leqslant \exp\{\tfrac{1}{2}\mathrm{Tr}[z(A_{q\phi}+A^*_{q\phi})+z^2|A_{q\phi}|^2]\} \end{aligned} \quad (5.10)$$

The inequality follows directly from

$$\ln(1 + x) \leqslant x \qquad \text{for } -1 < x < \infty$$

and the spectral theorem. We now notice that

$$\operatorname{Tr} A_{q\phi}^* = \operatorname{Tr} A_{-q\phi} = \int_\Lambda d^\nu x\, A_{-q\phi}(x, x)$$

$$\operatorname{Tr}(A_{q\phi} A_{q\phi}^*) = \int_\Lambda d^\nu x \int_\Lambda d^\nu y\, |A_{q\phi}(x, y)|^2$$

Therefore, using (5.1) and (5.2), we have

$$\operatorname{Tr}(A_{q\phi} + A_{q\phi}^*) \leqslant [\exp(\tfrac{1}{2}\beta q^2 K)](2\pi m/\beta)^{\nu/2}|\Lambda|$$

$$\operatorname{Tr}(A_{q\phi} A_{q\phi}^*) \leqslant [\exp(\beta q^2 K)](\pi m/\beta)^{\nu/2}|\Lambda|$$

so that

$$|\det(1 + zA_{q\phi})| \leqslant \exp(\{z[\exp(\tfrac{1}{2}\beta q^2 K)](2\pi m/\beta)^{\nu/2} + \tfrac{1}{2}z^2[\exp(\beta q^2 K)](\pi m/\beta)^{\nu/2}\}|\Lambda|) \tag{5.11}$$

Notice that from (5.10) and the Schwarz inequality for $\langle - \rangle_{\mathbb{V}}$ it follows that

$$\Xi_{\Lambda,-1}(\beta, z)^2 \leqslant \langle \exp z[\operatorname{Tr}(A_{q\phi} + A_{-q\phi})]\rangle_{\mathbb{V}} \langle \exp z^2 \operatorname{Tr}|A_{q\phi}|^2]\rangle_{\mathbb{V}} \tag{5.12}$$

The first term on the rhs of (5.12) is the partition function of a charge-conjugation-invariant quantum mechanical system with "Boltzmann statistics." Estimates (5.10)–(5.12) are very crude (far from being useful when the potential V has local singularities), but suffice for the purposes of this paper.

From now on we study charge-conjugation-invariant two-component systems, as in Theorem 4.1′, unless stated otherwise. It is assumed that the activity $z > 0$ is such that

$$\Xi_{\Lambda,\epsilon}(\beta, z) = \langle \exp \mathbb{C}_{\Lambda,\epsilon}(\beta, z; q\phi)\rangle_{\mathbb{V}}$$

is finite [e.g., $0 < z < \exp(-\beta q^2 K/2)$ for $\epsilon = 1$, and $0 < z < \infty$ for $\epsilon = -1$, $K < \infty$].

By (4.14), (4.15), and (4.23)

$$\mathbb{C}_{\Lambda,\epsilon}(\beta, z; q\phi) = \sum_{j=1}^{\infty} \frac{\epsilon^{j-1}}{j} z^j \int_\Lambda d^\nu x \int_\Omega P_{m,\Lambda}^{j\beta}(x, x; d\omega) \times \prod_{k=0}^{j-1} \exp\left[q^2/2 \int_0^\beta d\tau\, V(\omega(\tau + k\beta), \omega(\tau + k\beta))\right] \times \cos\left[q \sum_{k=0}^{j-1} \int_0^\beta d\tau\, \phi(\omega(\tau + k\beta), \tau)\right] \tag{5.13}$$

i.e., $\mathbb{C}_{\Lambda,\epsilon}(\beta, z; q\phi)$ is of the form

$$\sum_{j=1}^{\infty} \int d\rho^{\epsilon}_{\Lambda,j}(\xi) \cos[\phi(l^{(j)}_{\xi})] \tag{5.14}$$

with

$$|d\rho^{-1}_{\Lambda,j}| = d\rho^{1}_{\Lambda,j} \qquad \text{for all } j \tag{5.15}$$

[compare to Section 3, (3.3)]. In particular $\mathbb{C}_{\Lambda,\epsilon}$ is real-valued and *even* in ϕ.

Theorem 5.1 (Upper bounds). For $\epsilon = \pm 1$, $z > 0$,

$$\begin{aligned} &|\rho_{\Lambda,\epsilon}(\beta, z; (x)_N(x')_M; (y)_N(y')_M| \\ &\qquad \leqslant \delta_1^{(N)}(\tilde{\rho}(\beta, z, q; x_i, y_j))\, \delta_1^{(M)}(\tilde{\rho}(\beta, z, q; x_i', y_j')) \end{aligned}$$

with

$$\tilde{\rho}(\beta, z, q; x, y) = \left(\frac{2\pi m}{\beta}\right)^{\nu/2} \sum_{j=1}^{\infty} \frac{[z \exp(\beta q^2 K/2)]^j}{j^{\nu/2}} \exp\left[-\frac{|x-y|^2}{2j\beta}\right]$$

for $0 < z < \exp(-\beta q^2 K/2)$ [see (5.7)].

Proof. By (5.13)

$$\langle - \rangle_{\Lambda,\epsilon}(\beta, z) = \Xi_{\Lambda,\epsilon}(\beta, z)^{-1} \langle - \exp \mathbb{C}_{\Lambda,\epsilon}(\beta, z; q\phi)\rangle_{\mathbb{V}}$$

is the expectation given by a positive probability measure. Hence

$$\begin{aligned} |\rho_{\Lambda,\epsilon}(\beta, z; (x)_N(x')_M; (y)_N(y')_M)| &\leqslant \left|\delta_\epsilon^{(N)}\left(z \frac{\partial}{\partial z} \rho_{\Lambda,\epsilon}(\beta, z; x_i, y_j; q\phi)\right)\right| \\ &\quad \times \left|\delta_\epsilon^{(M)}\left(z \frac{\partial}{\partial z} \rho_{\Lambda,\epsilon}(\beta, z; x_i', y_j'; -q\phi)\right)\right| \end{aligned}$$

Next

$$\left|\delta_\epsilon^{(N)}\left(z \frac{\partial}{\partial z} \rho_{\Lambda,\epsilon}(\beta, z; x_i, y_j; \pm q\phi)\right)\right| \leqslant \delta_1^{(N)}\left(z\left|\frac{\partial}{\partial z} \rho_{\Lambda,1}(\beta, z; x_i, y_j; \pm q\phi)\right|\right)$$

as one easily deduces from (4.19) (definition of $\delta_\epsilon^{(N)}$) and (4.14) [definition of $\rho_{\Lambda,\epsilon}(m, z; x, y; \pm q\phi)$].

Use of formula (4.28) and inequality (5.6) completes the proof. ∎

Theorem 5.2 ($|\mathrm{RDM}_{-1}| \leqslant \mathrm{RDM}_{+1}$). For charge-conjugation-invariant, two-component systems

$$|\rho_{\Lambda,-1}(\beta, z; (x)_N(x')_M; (y)_N(y')_M)| \leqslant \rho_{\Lambda,+1}(\beta, z; (x)_N(x')_M; (y)_N(y')_M)|$$

i.e., the Bose RDM dominates the absolute values of the Fermi RDM pointwise.

Proof. By definition [see (4.19)]

$$\delta_\epsilon^{(N)}\left(z\frac{\partial}{\partial z}\rho_{\Lambda,\epsilon}(\beta, z; x_i, y_j; \pm q\phi)\right)$$
$$= \sum_{\pi\in S_N}\epsilon^{\sigma(\pi)}\prod_{j=1}^{N} z\frac{\partial}{\partial z}\rho_{\Lambda,\epsilon}(\beta, z; x_j, y_{\pi(j)}; \pm q\phi)$$

where

$$z\frac{\partial}{\partial z}\rho_{\Lambda,\epsilon}(\beta, z; x, y; \pm q\phi) = \sum_{j=1}^{\infty}\epsilon^{j-1}z^j\int_\Omega P_{m,\Lambda}^{j\beta}(x, y; d\omega)$$
$$\times\exp\left[\frac{1}{2}\int_0^{j\beta} d\tau\, V(\omega(\tau), \omega(\tau))\right]$$
$$\times\exp\left[\pm i\sum_{k=0}^{j-1} q\int_0^\beta d\tau\,\phi(\omega(\tau+k\beta), \tau)\right]$$

[see (4.14) or (4.28), (4.26)].

Thus the *even* part of

$$\delta_{-1}^{(N)}\left(z\frac{\partial}{\partial z}\rho_{\Lambda,-1}(\beta, z; x_i, y_j; q\phi)\right)\delta_{-1}^{(M)}\left(z\frac{\partial}{\partial z}\rho_{\Lambda,-1}(\beta, z; x_i', y_j'; -q\phi)\right)$$

is of the form

$$\sum_k\int d\lambda_k(m^{(k)})\cos[\phi(m^{(k)})+\theta^{(k)}] \tag{5.16}$$

where $\{d\lambda_k\}_{k=1}^{\infty}$ are positive measures on appropriate function spaces, and $\theta^{(k)}$ are phases ($=0$ or π). This is to be compared with the even part of

$$\delta_1^{(N)}\left(z\frac{\partial}{\partial z}\rho_{\Lambda,1}(\,\cdot\,; x_i, y_j; \,\cdot\,)\right)\delta_1^{(M)}\left(z\frac{\partial}{\partial z}\rho_{\Lambda,1}(\,\cdot\,; x_i', y_j'; \,\cdot\,)\right)$$

which has the form

$$\sum_k\int d\lambda_k'(m^{(k)})\cos[\phi(m^{(k)})+\theta'^{(k)}] \tag{5.17}$$

and by inspection we see that

$$d\lambda_k' = d\lambda_k \qquad\text{and}\qquad \theta'^{(k)} = 0 \qquad\text{for all } k \tag{5.18}$$

Recalling properties (5.14) and (5.15) of $\mathbb{C}_{\Lambda,\epsilon}$, we thus conclude that the correlation inequality of Theorem 3.2 can be used here. A special case of it is

$$\langle\cos\phi(m^{(k)})\rangle_{\Lambda,1}(\beta, z) \geqslant \langle\cos[\phi(m^{(k)})+\theta''^{(k)}]\rangle_{\Lambda,-1}(\beta, z)$$

with $\theta''^{(k)} = \theta^{(k)}$ or $\theta^{(k)} + \pi$. Hence

$$\langle \cos \phi(m^{(k)}) \rangle_{\Lambda,1}(\beta, z) \geqslant |\langle \cos[\phi(m^{(k)}) + \theta^{(k)}] \rangle_{\Lambda,-1}(\beta, z)| \tag{5.19}$$

Since $\langle - \rangle_{\Lambda,\pm 1}(\beta, z)$ is even in ϕ, the proof now follows directly from (5.16)–(5.18). ■

Remarks. 1. Using Theorem 4.1″ and (4.24′), Section 4.2, the extension of Theorems 5.1 and 5.2 to systems of particles with spin is straightforward. Moreover, we can see generalizations to ITGFs; see Section 4.3.

2. For Fermi statistics, the upper bound on the RDMs given in Theorem 5.1 is poor and fairly uninteresting. Uniform upper bounds on RDMs (or ITGFs), smeared out with test functions, follow from the boundedness of fermion creation and annihilations operators (a consequence of the canonical anticommutation relations), as is well known.

3. In the same sense as Theorem 5.1, Theorem 5.2, i.e., domination of Fermi RDMs by Bose RDMs, may be regarded as an uninteresting and physically obvious statement. We still feel that it is somewhat remarkable that it is true mathematically.

Theorem 5.3 (Lower bounds):

$$\rho_{\Lambda,1}(\beta, z; (x)_N(x')_M; (y)_N(y')_M) \geqslant \left\langle \delta_1^{(N)}\left(z \frac{\partial}{\partial z} \rho_{\Lambda,1}(\beta, z; x_i, y_j; q\phi)\right)\right.$$

$$\left. \times\, \delta_1^{(M)}\left(z \frac{\partial}{\partial z} \rho_{\Lambda,1}(\beta, z; x_i, y_j; -q\phi)\right)\right\rangle_V$$

Proof. In view of (5.14), (5.15), (5.17), and (5.18) this is a direct consequence of the correlation inequality in Theorem 3.1. ■

It should be noted that Theorem 5.3 can be proven more directly by using Jensen's inequality and a trivial inequality on permanents at the right places, rather than the ϕ-functional integral and a correlation inequality.

Next, we show that the rhs of the inequality in Theorem 5.3 diverges if z is large enough and Λ tends to $\mathbb{R}^\nu$, provided the potential V falls off sufficiently rapidly. We are indebted to M. Campanino for suggesting to us the main idea in the following argument.

First, we consider the two-point RDM, $\rho_{\Lambda,1}(\beta, z; x, y)$ [the case of a general $(2N, 2M)$-point RDM being similar]. By Theorem 5.3 we have

$$\rho_{\Lambda,1}(\beta, z; x, y) \geqslant \left\langle z \frac{\partial}{\partial z} \rho_{\Lambda,1}(\beta, z; x, y; \pm q\phi) \right\rangle_{\mathbb{V}}$$
$$= \sum_{j=1}^{\infty} z^j \int_{\Omega} P_{m,\Lambda}^{j\beta}(x, y; d\omega)$$
$$\times \exp\left[-(q^2/2) \sum_{\substack{k,l=0 \\ k \neq l}}^{j-1} \int_0^{\beta} d\tau \, V(\omega(\tau + k\beta), \omega(\tau + l\beta)) \right] \tag{5.20}$$

We propose to show that, for z large enough, the rhs of (5.20) diverges, as $\Lambda \nearrow \mathbb{R}^{\nu}$. Let e_1 be the unit vector in the direction of $y - x$ (resp. the unit vector in the positive 1-direction if $y = x$).

Let R be some positive number. We define a sequence of points

$$\xi_k' = x + \tfrac{1}{2}kRe_1, \qquad k = 0, 1, 2,... \tag{5.21}$$

Define k_0 by the property that

$$\min_k |y - \xi_k'| = |y - \xi_{k_0}'|$$

Since $|\xi_{k+1}' - \xi_k'| = \frac{1}{2}R$,

$$|y - \xi_{k_0}'| \leqslant \tfrac{1}{4}R \tag{5.22}$$

Given an integer $j > 0$, define $k_1 \geqslant k_0$ by the equation

$$k_0 - 1 + 2(k_1 - k_0 + 1) = j - 1 \text{ or } j$$

i.e.,

$$k_1 = [(j + k_0 - 1)/2] \tag{5.23}$$

where $[a]$ is the largest integer $\leqslant a$. If $j < k_0$, then k_1 is not defined. We now define a sequence of points $\{\xi_k\}_{k=0}^{j-1}$ as follows (Fig. 1):

$$\text{for } k_1 = (j + k_0 - 1)/2: \quad \begin{array}{ll} \xi_k = \xi_k' & \text{for } k \leqslant k_1 \\ \xi_k = \xi_{2k_1-k}' & \text{for } k_1 < k \leqslant j - 1 \end{array} \tag{5.24}$$

$$\text{for } k_1 = (j + k_0)/2 - 1: \quad \begin{array}{ll} \xi_k = \xi_k' & \text{for } k \leqslant k_1 \\ \xi_k = \xi_{2k_1-k+1}' & \text{for } k_1 < k \leqslant j - 1 \end{array} \tag{5.25}$$

Let S_R^k be the ball of radius R centered at ξ_k. Let j_Λ be such that

$$S_R^k \subset \Lambda \qquad \text{for } k = 1,..., j - 1, \quad \text{all } j \leqslant j_\Lambda \tag{5.26}$$

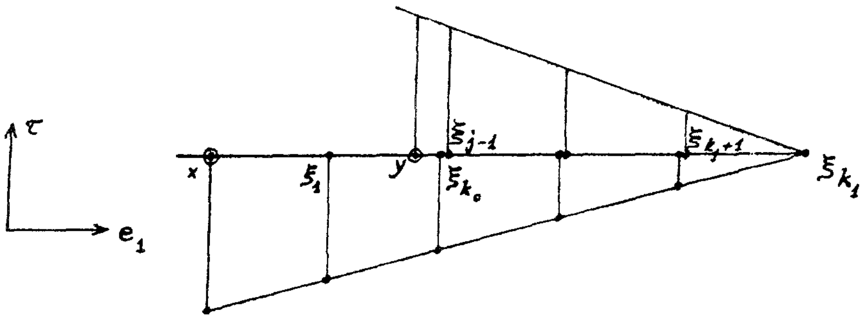

Fig. 1

Then, for $k_0 \leqslant j \leqslant j_\Lambda$

$$P^{j\beta}_{m,\Lambda}(x, y; d\omega) = \chi^{j\beta}_{\Lambda}(\omega) P^{j\beta}_{m}(x, y; d\omega) \geqslant \left\{\sum_{n=0}^{j-1} \chi^{(n)}_{S^n_R}(\omega)\right\} P^{j\beta}_{m}(x, y; d\omega) \quad (5.27)$$

where $\chi^{(n)}_{S}(\omega)$ is the characteristic function of

$$\{\omega : \omega(\tau) \in S \quad \text{for } n\beta \leqslant \tau \leqslant (n+1)\beta\}$$

Next,

$$\left[\sum_{l=0, l\neq k}^{j-1} \int_0^\beta d\tau\, V(\omega(\tau + k\beta), \omega(\tau + l\beta))\right] \prod_{n=0}^{j-1} \chi^{(n)}_{S^n_R}(\omega)$$

$$\leqslant \sum_{l=0, l\neq k}^{j-1} \max_{x \in S^k_R, y \in S^l_R} \beta |V(x, y)| \quad (5.28)$$

We now assume that

$$c(V) \equiv \max_k \sum_{l=1}^{\infty} \max_{x \in S'^k_R, y \in S'^l_R} |V(x, y)| \quad (5.29)$$

is *finite*. Here S'^k_R is the ball of radius R centered at ξ_k', $k = 0, 1, 2, \ldots$. Condition (5.29) constrains the falloff of the potential V as $|x - y| \to \infty$. It is fulfilled, e.g., if $V(x, y) = W(x - y)$, where W is a bounded, continuous function on $\mathbb{R}^\nu$ with

$$|W(x)| \leqslant \text{const} \cdot |x|^{-1-\epsilon} \qquad \text{for some } \epsilon > 0 \quad (5.30)$$

Under these hypotheses on V the proof of (5.29) follows immediately from (5.21).

By (5.20) and (5.27)–(5.30)

$$\rho_{\Lambda,1}(\beta, z; x, y) \geqslant \sum_{j=k_0}^{j_\Lambda} z^j e^{-jq^2\beta c(V)} \int_\Omega \prod_{n=0}^{j-1} \chi^{(n)}_{S^n_R}(\omega) P^{j\beta}_{m}(x, y; d\omega) \quad (5.31)$$

The integral on the rhs of (5.31) is easy to estimate from below: Define

$$\alpha = \min_{\xi \in \tilde{S}_{R/2}} \int_{\tilde{S}_{R/2}} d^\nu \eta \int_\Omega P^\beta_{m,S^1_R}(\xi, \eta; d\omega) \tag{5.32}$$

where $\tilde{S}_{R/2}$ is the sphere of radius $R/2$ centered at $x + \frac{1}{4}Re_1$; moreover,

$$\alpha' = \min_{\xi \in \tilde{\tilde{S}}_{R/2}} \int_\Omega P^\beta_{m,S^{j-1}_R}(\xi, y; d\omega) \tag{5.33}$$

where $\tilde{\tilde{S}}_{R/2}$ is the sphere centered at $1/2(\xi_{j-1} + \xi_{j-2})$. Since $\int_\Omega P^\beta_{m,S}(\xi, \eta; d\omega)$ is the kernel of $\exp[(\beta/2m)\Delta^S]$, where Δ^S is the Laplacian with 0-Dirichlet data at ∂S,

$$\alpha > 0 \text{ and } \alpha' > 0 \qquad \text{for all } R > 0 \tag{5.34}$$

From (5.31), the Markov property, (5.32), and (5.33) we deduce

$$\rho_{\Lambda,1}(\beta, z; x, y) \geqslant \frac{\alpha'}{\alpha} \sum_{j=k_0}^{j_\Lambda} (z\alpha e^{-q^2\beta c(V)})^j \tag{5.35}$$

As $\Lambda \nearrow \mathbb{R}^\nu$, j_Λ tends to $+\infty$, and the rhs of (5.35) approaches

$$(z\alpha e^{-q^2\beta c(V)})^{k_0}/[1 - z\alpha e^{-q^2\beta c(V)}] \tag{5.36}$$

Clearly (5.36) and hence $\rho_{\Lambda,1}(\beta, z; x, y)$ *diverge* when

$$z > \min_R (\alpha^{-1} e^{q^2\beta c(V)}), \qquad \text{for all } x \text{ and } y \tag{5.37}$$

It is not hard to extend the above arguments to the $(2N, 2M)$-point RDMs: If $(x)_N = (x')_M$ and $(y)_N = (y')_M$ one can simply use the Hölder inequality with respect to the expectation $\langle - \rangle_{\Lambda,1}(\beta, z)$ to show that $\rho_{\Lambda,1}(\beta, z; (x)_N(x)_N; (y)_N(y_N))$ diverges if $\rho_{\Lambda,1}(\beta, z; x, y)$ diverges. For general RDMs and

$$z > \min_R (\alpha^{-1} e^{q^2\beta c(V)})\, \delta^{\beta(N+M-1)} \qquad \text{for some } \delta > 1$$

$\rho_{\Lambda,1}(\beta, z; (x)_N(x')_M; (y)_N, (y')_M)$ has a *divergent* lower bound. (The details of this generalization, as well as estimates on δ, are rather straightforward and are left to the reader.) Finally we remark that spin can be included, as is obvious from Theorem 4.1″.

We summarize as follows:

Theorem 5.4. For a charge-conjugation-invariant system of two species of bosons of charge q and spin S interacting via a (spin-independent) two-body potential V with the property that $c(V)$ defined in (5.29) is finite, the RDMs $\rho_{\Lambda,1}(\beta, z; (r)_N(r')_M; (\bar{r})_N(\bar{r}')_M)$ diverge, for arbitrary points $(x)_N$, $(x')_M$, $(y)_N$, $(y')_M$, provided z is large enough (depending on β, V,...).

Remark. This result suggests that charge-conjugation-invariant, two-component Bose gases must have Bose–Einstein condensation when the density is large. Physically speaking, we expect oppositely charged particles to form neutral "molecules" at large density. But our results are clearly not even quite a beginning of a rigorous theory of Bose–Einstein condensation. An interesting open problem in an attempt toward a rigorous theory of BE condensation is: Prove an upper bound (*infrared bound*) on $\hat{\rho}_{\Lambda,1}(\beta, z; k, -k)$ for small momenta $k \neq 0$, e.g., in terms of the ideal-Bose-gas two-point RDM.

5.2. Existence of the Thermodynamic Limit: Bose Statistics

Let

$$G_j(\omega) \equiv \prod_{k=0}^{j-1} \exp\left[q^2/2 \int_0^\beta d\tau\, V(\omega(\tau + k\beta), \omega(\tau + k\beta))\right] \tag{5.38}$$

Recall that in the notation of (5.13), (5.14)

$$\begin{aligned}\mathbb{C}_{\Lambda,1}(\beta, z; q\phi) &= \sum_{j=1}^{\infty} \frac{z^j}{j} \int d^\nu x \int_\Omega P^{j\beta}_{m,\Lambda}(x, x; d\omega) G_j(\omega) \\ &\quad \times \cos\left[\sum_{k=0}^{j-1} q \int_0^\beta d\tau\, \phi(\omega(\tau + k\beta), \tau)\right] \\ &\equiv \sum_{j=1}^{\infty} \int d\rho^1_{\Lambda,j}(\xi) \cos[\phi(l^{(j)}_\xi)]\end{aligned} \tag{5.39}$$

i.e., $d\rho^1_{\Lambda,j}$ is given by $\int d^\nu x \int_\Omega G_j(\omega) P^{j\beta}_{m,\Lambda}(x, x; d\omega)$, with

$$P^{j\beta}_{m,\Lambda}(x, x; d\omega) = \chi^{j\beta}_\Lambda(\omega) P^{j\beta}_m(x, x; d\omega)$$

By definition, $\chi^{j\beta}_\Lambda(\omega)$ is pointwise monotone increasing in Λ, i.e., if $\Lambda' \supseteqq \Lambda$, then

$$\chi^{j\beta}_{\Lambda',\Lambda}(\omega) \equiv \chi^{j\beta}_{\Lambda'}(\omega) - \chi^{j\beta}_\Lambda(\omega) \geqslant 0$$

so that

$$d\rho^1_{\Lambda',\Lambda,j} \equiv d\rho^1_{\Lambda',j} - d\rho^1_{\Lambda,j} \geqslant 0 \tag{5.40}$$

Thus, for $\Lambda' \supset \Lambda$,

$$\mathbb{C}_{\Lambda',1}(\beta, z; q\phi) = \mathbb{C}_{\Lambda,1}(\beta, z; q\phi) + \sum_{j=1}^{\infty} \int d\rho^1_{\Lambda',\Lambda,j}(\xi) \cos \phi(l^{(j)}_\xi) \tag{5.41}$$

for some positive measures $d\rho^1_{\Lambda',\Lambda,j}$.

Theorem 5.5 (Existence of the thermodynamic limit). Under the hypotheses of Theorem 5.1 [i.e., for $0 < z < z_c$ with $z_c \geqslant \exp(-\beta q^2 K/2)$] and $\epsilon = 1$, i.e., Bose statistics,

$$\lim_{\Lambda \nearrow \mathbb{R}^\nu} \rho_{\Lambda,1}(\beta, z; (x)_N(x')_M; (y)_N(y')_M) \equiv \rho_1(\beta, z; (x)_N(x')_M; (y)_N(y')_M)$$

exists and has the same spatial symmetries as the Hamiltonian. In particular, if V is invariant under Euclidean motions, then so are the RDMs ρ_1, for all N and M.

Proof. By Theorem 5.1, it is enough to prove that

$$\rho_{\Lambda,1}(\beta, z; (x)_N(x')_M; (y)_N(y')_M)$$

is monotone increasing in Λ. Theorem 5.5 then follows by standard arguments; see Ref. 22 and I.

As asserted in (5.17) and (5.18),

$$\delta_1^{(N)}\left(z \frac{\partial}{\partial z} \rho_{\Lambda,1}(\cdot\,; x_i, y_j; q\phi)\right) \delta_1^{(M)}\left(z \frac{\partial}{\partial z} \rho_{\Lambda,1}(\cdot\,; x_i', y_j'; q\phi)\right)$$

is of the form

$$\sum_k \int d\lambda_{k,\Lambda}(m^{(k)}) \cos \phi(m^{(k)}) \tag{5.42}$$

and it is shown by the same arguments that we used to prove (5.40) that the measures $d\lambda_{k,\Lambda}$ are *monotone increasing* in Λ.

We now define

$$\langle - \rangle(s; \beta, z) = \Xi(s; \beta, z)^{-1} \times \left\langle - \exp\left[\mathbb{C}_{\Lambda'}(\beta, z; q\phi) + s \sum_{j=1}^{\infty} \int d\rho^1_{\Lambda'',\Lambda',j}(\xi) \cos \phi(l_\xi^{(j)}) \right] \right\rangle_{\nu} \tag{5.43}$$

where $\Xi(s; \beta, z)$ is the obvious normalization factor.

From (5.43) and Theorem 3.1(i) it follows that

$$\int d\lambda_{k,\Lambda}(m^{(k)}) \langle \cos \phi(m^{(k)}) \rangle (s; \beta, z) \tag{5.44}$$

is monotone increasing in Λ. Furthermore,

$$\frac{\partial}{\partial s} \langle \cos \phi(m^{(k)}) \rangle (s; \beta, z) = \sum_{j=1}^{\infty} \int d\rho^1_{\Lambda'',\Lambda',j}(\xi) \langle \cos \phi(m^{(k)}); \cos \phi(l_\xi^{(j)}) \rangle (s; \beta, z)$$

and the rhs is *nonnegative*, by Theorem 3.1(ii). Integrating over s from 0 to 1 then shows that $\langle \cos \phi(m^{(k)}) \rangle_{\Lambda,1}(\beta, z)$ is monotone increasing in Λ. This property together with (5.44) yields monotonicity of the RDMs in Λ. ∎

Theorem 5.6 (Monotonicity in z and V). Under the same hypotheses and for arbitrary $\Lambda \subseteq \mathbb{R}^\nu$:

(i) $\rho_{\Lambda,1}(\beta, z; (x)_N(x')_M; (y)_N(y')_M)$ is monotone increasing in z.

(ii) If V is translation invariant with $V(0) \equiv K < \infty$, and for $\bar{z} = e^{-\beta q^2 K/2} z$, $\rho_{\Lambda,1}(\beta, \bar{z}; (x)_N(x')_M; (y)_N(y')_M)$ decreases when V increases as a quadratic form.

Proof. 1. This follows from Theorem 3.1(ii) by the arguments used in the proof of Theorem 5.5; see also I, Theorem QM, §1.

2. The proof of this is identical to that of Corollary 3.2, (1), §3 of paper I, except for notational complications [the basic ingredients used are Theorem 3.1(ii) and the fact that the covariance $\mathbb{V}$ of ϕ increases if V increases in the quadratic form sense].

Generalizations. Theorems 5.5 and 5.6 also hold for bosons with spin. This is checked with the help of Theorem 4.1″, Section 4.2. Moreover, one can apply the arguments used in this section to general ITGFs, with identical conclusions. To see this, one makes use of the machinery outlined in the last part of Section 4.3. By a general reconstruction theorem, the ITGFs in the thermodynamic limit uniquely determine a β-KMS state and the dynamics of the infinite system in *thermal equilibrium*.

5.3. Electromagnetic Fields

The coupling of a quantum mechanical particle with electric charge e to a classical or quantized electromagnetic vector potential $\mathbf{A} = (A_1, ..., A_\nu)$ is achieved by the usual minimal substitution

$$\partial_j \equiv \partial/\partial x^j \mapsto \partial_j - ieA_j, \qquad \Delta^\Lambda \mapsto \Delta_{\mathbf{A}}{}^\Lambda = \sum_{j=1}^{\nu} (\partial_j - ieA_j)^*(\partial_j - ieA_j) \quad (5.45)$$

The kinetic energy operator $T_\Lambda^{(M,N)}$ defined in (2.2) is replaced by

$$T_{\Lambda,\mathbf{A}}^{(M,N)} = -\sum_{i=1}^{M} (1/2m_1)\Delta_{i,\mathbf{A}}^\Lambda - \sum_{j=1}^{N} (1/2m_2)\Delta_{j,\mathbf{A}}'^\Lambda \quad (5.46)$$

The total Hamiltonian is given by the previous expression, except that $T_\Lambda^{(M,N)}$ is replaced by $T_{\Lambda,\mathbf{A}}^{(M,N)}$. From now on we impose the *Coulomb* (radiation) *gauge* on $\mathbf{A}$, i.e.,

$$A_0 \equiv 0, \qquad (\nabla \cdot \mathbf{A})(x, t) = \sum_{i=1}^{\nu} \partial_i A_i(x, t) = 0 \quad (5.47)$$

Next, we recall the path space formula for $\exp\{-\beta[-(1/2m)\Delta_{\mathbf{A}}{}^{\Lambda} + W]\}$, where W is a bounded one-particle potential. The integral kernel of this operator is given by the following modified Feynman–Kac formula:

$$\int_{\Omega} P^{\beta}_{m,\Lambda}(x, y; d\omega) \exp\left[ie \sum_{j=1}^{\nu} \int A_j(\omega(\tau), \tau)\, d\omega^j(\tau)\right] \times \exp\left[-\int_0^{\beta} d\tau\, W(\omega(\tau))\right] \tag{5.48}$$

where $\int A_j(\omega(\tau), \tau)\, d\omega^j(\tau)$ is defined as an Ito stochastic integral; the definition is unambiguous, thanks to the Coulomb gauge condition (5.47). See, e.g., Ref. 29. [A convenient way of deriving and interpreting (5.48) is also provided by the lattice approximation: replace $\mathbb{R}^{\nu}$ by $a\mathbb{Z}^{\nu}$, establish (5.48) on the lattice $a\mathbb{Z}^{\nu}$, and then pass to the limit $a \searrow 0$. This program is carried out, e.g., in Ref. 4.]

If the external vector potential is classical and stationary, then

$$\mathbf{A}(\omega(\tau), \tau) = \mathbf{A}(\omega(\tau))$$

is an $\mathbb{R}^{\nu}$-valued function on Ω which does *not* explicitly depend on τ. If $\mathbf{A}$ is the quantized vector potential in the Coulomb gauge, then $\mathbf{A}(x, \tau)$ is interpreted as the corresponding Euclidean field with periodic boundary conditions at $\tau = 0, \beta$. It is a *Gaussian*, $\mathbb{R}^{\nu}$-valued, divergence-free random field with mean 0 and divergence $D^{\beta}_{ij}(x - x', \tau - \tau')$, the transverse Euclidean ($\equiv$ imaginary time) propagator of the free electromagnetic field, which is periodic in $\tau - \tau'$ with period β. As is well known, this corresponds to an inverse temperature β equilibrium state of the free em field. The corresponding Gaussian measure ("the law of $\mathbf{A}$") is denoted $dm^{\beta}(\mathbf{A})$.

In order to avoid all problems with ultraviolet renormalizations, an ultraviolet (high-frequency) cutoff in the spatial directions is imposed upon $\mathbf{A}$, with the effect that

$$\int dm^{\beta}(\mathbf{A})\, A_i(x, \tau) A_j(x', \tau') = D^{\beta}_{ij}(x - x', \tau - \tau') \tag{5.49}$$

is regular at (0, 0). In this case all subsequent formulas of this section are free of Wick ordering (of powers of $\mathbf{A}$) and of counterterms, without ultraviolet divergences arising.

We now define the analog of the one-particle operator $A_{q\phi} \equiv A^{\beta}_{m,q\phi}$ introduced in (4.24), (4.25), Section 4.2:

$$A_{q\phi,\mathbf{A}}(x, y) = \int_{\Omega} P^{\beta}_{m,\Lambda}(x, y; d\omega) \exp\left[q^2/2 \int_0^{\beta} d\tau\, V(\omega(\tau), \omega(\tau))\right] \times \exp\left[iq \int_0^{\beta} d\tau\, \phi(\omega(\tau), \tau) + \sum_{l=1}^{\nu} ie \int A_l(\omega(\tau), \tau)\, d\omega^l(\tau)\right] \tag{5.50}$$

We then set

$$\begin{aligned}\mathbb{C}_{\Lambda,\epsilon}(\beta, z; q\phi; \mathbf{A}) &= -\epsilon\{\mathrm{Tr}\ln(1-\epsilon z A_{q\phi,\mathbf{A}}) + \mathrm{Tr}\ln(1-\epsilon z A_{-q\phi,-\mathbf{A}})\}\\ &= \sum_{j=1}^{\infty}\frac{\epsilon^{j-1}}{j}z^j\int d^\nu x\int_\Omega P^{j\beta}_{m,\Lambda}(x,x;d\omega)G_j(\omega)\\ &\quad\times\cos\left\{\sum_{k=0}^{j-1}\left[q\int_0^\beta d\tau\,\phi(\omega(\tau+k\beta),\tau)\right.\right.\\ &\quad\left.\left.+\sum_{l=1}^{\nu}e\int A_l(\omega(\tau+k\beta,\tau)\,d\omega^l(\tau+k\beta)\right]\right\}\\ &\equiv\sum_{j=1}^{\infty}\epsilon^{j-1}\int d\rho_{\Lambda,j}(\xi)\cos[\phi(l^{(j)}_\xi)+A(h^{(j)}_\xi)]\end{aligned}\tag{5.51}$$

where $G_j(\omega)$ is the Wick ordering factor defined in (5.38), and the last expression is a short hand for the complicated third expression.

Next, let

$$\Xi_{\Lambda,\epsilon}(\beta, z; \mathbf{A}) = \langle\exp\mathbb{C}_{\Lambda,\epsilon}(\beta, z; q\phi; \mathbf{A})\rangle_{\mathbb{V}}\tag{5.52}$$

$$\begin{aligned}z\frac{\partial}{\partial z}\rho_{\Lambda,\epsilon}(\beta, z; x, y; q\phi; \mathbf{A}) &= z\{(1-\epsilon z A_{q\phi,\mathbf{A}})^{-1}A_{q\phi,\mathbf{A}}\}(x,y)\\ &= z\frac{\partial}{\partial z}\{-\epsilon\ln(1-\epsilon z A_{q\phi,\mathbf{A}})\}(x,y)\end{aligned}\tag{5.53}$$

[see (4.27), (4.28)]. The correlation functions in an external vector potential **A** are then given by

$$\begin{aligned}&\rho_{\Lambda,\epsilon}(\beta, z; (x)_N,\ldots,(y')_M; \mathbf{A})\\ &\quad= \Xi_{\Lambda,\epsilon}(\beta, z; \mathbf{A})^{-1}\left\langle\delta^{(N)}_\epsilon\left(z\frac{\partial}{\partial z}\rho_{\Lambda,\epsilon}(\beta, z; x_i, y_j; q\phi; \mathbf{A})\right)\right.\\ &\qquad\left.\times\,\delta^{(M)}_\epsilon\left(z\frac{\partial}{\partial z}\rho_{\Lambda,\epsilon}(\beta, z; x_i', y_j'; q\phi; \mathbf{A})\right)\exp\mathbb{C}_{\Lambda,\epsilon}(\beta, z; q\phi; \mathbf{A})\right\rangle_{\mathbb{V}}\end{aligned}\tag{5.54}$$

and the correlation functions of the fully quantized system by

$$\begin{aligned}\rho_{\Lambda,\epsilon}(\beta, z; (x)_N,\ldots,(y')_M; \mathbf{f}) &= \Xi^{\mathrm{tot}}_{\Lambda,\epsilon}(\beta, z)^{-1}\int dm^\beta(\mathbf{A})\{\exp[i\mathbf{A}(\mathbf{f})]\}\Xi_{\Lambda,\epsilon}(\beta, z; \mathbf{A})\\ &\quad\times\rho_{\Lambda,1}(\beta, z; (x)_N,\ldots,(y')_M; \mathbf{A})\end{aligned}\tag{5.55}$$

with

$$\Xi^{\mathrm{tot}}_{\Lambda,\epsilon}(\beta, z) = \int dm^\beta(\mathbf{A})\Xi_{\Lambda,\epsilon}(\beta, z; \mathbf{A})\tag{5.56}$$

We now discuss the following three problems:

(I) Diamagnetic inequality for partition function.[29]

(II) Diamagnetic inequality for RDMs.

(III) Existence of the thermodynamic limit of the RDMs of non-relativistic quantum electrodynamics.

(I) We begin by recalling Simon's diamagnetic inequality,[29] i.e.,

$$\Xi_{\Lambda,1}(\beta, z; \mathbf{A}) \leqslant \Xi_{\Lambda,1}(\beta, z; \mathbf{0}) = \Xi_{\Lambda,1}(\beta, z) \tag{5.57}$$

It must be emphasized that (5.57) holds for general Bose systems, *without* the assumption of charge-conjugation invariance. In the formalism adopted in the present paper the proof of (5.57) proceeds as follows: notice that $A_{\pm q\phi, \pm \mathbf{A}}(x, y)$ defined in (5.50) is of positive type as a function of ϕ and $\mathbf{A}$. Thus $-\operatorname{Tr}\ln(1 - zA_{\pm q\phi,\pm\mathbf{A}})$ and consequently $\exp[-\operatorname{Tr}\ln(1 - zA_{\pm q\phi,\pm\mathbf{A}})]$ are of positive type in ϕ and $\mathbf{A}$. Since $\langle - \rangle_V$ is *Gaussian*,

$$\Xi_{\Lambda,1}(\beta, z; \mathbf{A}) \text{ is of positive type in } \mathbf{A} \tag{5.58}$$

(See Ref. 4, Section 5, for details concerning related arguments.) We emphasize that (5.58) really holds for general Bose systems of arbitrarily many species of bosons of arbitrary spin, as long as the spin of the particles is *not* coupled to the electromagnetic field. Clearly (5.58) implies Simon's inequality (5.57).

Since $dm^\beta(\mathbf{A})$ is *Gaussian* (i.e., of positive type),

$$\Xi^{\text{tot}}_{\Lambda,1}(\beta, z; \mathbf{A}_{\text{cl}}) \equiv \int dm^\beta(\mathbf{A})\, \Xi_{\Lambda,1}(\beta, z; \mathbf{A} + \mathbf{A}_{\text{cl}})$$

is of positive type in the classical, external field, $\mathbf{A}_{\text{cl}}$, so that

$$\Xi^{\text{tot}}_{\Lambda,1}(\beta, z; \mathbf{A}_{\text{cl}}) \leqslant \Xi^{\text{tot}}_{\Lambda,1}(\beta, z; \mathbf{0}) = \Xi^{\text{tot}}_{\Lambda,1}(\beta, z) \tag{5.59}$$

This says that *if the interactions of the spin of bosons with the electromagnetic field is neglected, then systems of arbitrarily many species of bosons of arbitrary spin react diamagnetically to an external electromagnetic field* $\mathbf{A}_{\text{cl}}$.

(II) Next, we prove a related result for the RDMs of charge-conjugation-invariant Bose systems.

Theorem 5.7 (Diamagnetism in RDMs). Assume charge-conjugation invariance. Then

(i) $$|\rho_{\Lambda,1}(\beta, z; (x)_N, \ldots, (y')_M; \mathbf{A})| \leqslant \rho_{\Lambda,1}(\beta, z; (x)_N, \ldots, (y')_M)$$

(ii) $$|\rho_{\Lambda,-1}(\beta, z; (x)_N, \ldots, (y')_M; \mathbf{f})| \leqslant \rho_{\Lambda,1}(\beta, z; (x)_N, \ldots, (y'); \mathbf{f})$$

Proof. (i) In view of (5.51), (5.50), and (5.53), result (i) reveals itself as a special case of Theorem 3.2. [The role of the phases α, $\{f\}$ in Theorem 3.2 is played by

$$\sum_{l=1}^{\nu} ie \int A_l(\omega(\tau + k\beta), \tau)\, d\omega^l(\tau + k\beta), \quad k = 0, 1, 2,...]$$

See also (5.17) and (5.18).

(ii) This is a straightforward generalization of Theorem 5.2 with a very similar proof, which we permit ourselves to leave to the reader. ■

We remark that Theorem 5.7(i) can be generalized in the same way as (5.57) is generalized to (5.59), and again spin can be included if not coupled to **A**. Next, suppose the electromagnetic field is quantized. Since $\Xi_{\Lambda,1}(\beta, z; \mathbf{A})$ is of *positive type* in **A**, it has the general form $\int d\rho_\Lambda(\mathbf{h}) \exp[i\mathbf{A}(\mathbf{h})]$ for some $d\rho_\Lambda \geqslant 0$. By Theorem 3.1(ii) we therefore have

$$\begin{aligned}\langle \cos \mathbf{A}(\mathbf{f}) \rangle_{\Lambda,1}(\beta, z) &\equiv \Xi_{\Lambda,1}^{tot}(\beta, z)^{-1} \int dm^\beta(\mathbf{A})\, [\cos \mathbf{A}(\mathbf{f})] \Xi_{\Lambda,1}(\beta, z; \mathbf{A}) \\ &\geqslant \int dm^\beta(\mathbf{A}) \cos \mathbf{A}(\mathbf{f})\end{aligned}$$

and since

$$x^2 = 2 \lim_{\epsilon \searrow 0} \epsilon^{-2}(\mathbf{1} - \cos \epsilon x)$$

then

$$\langle |\mathbf{A}(\mathbf{f})|^2 \rangle_{\Lambda,1}(\beta, z) \leqslant \int dm^\beta(\mathbf{A})\, |\mathbf{A}(\mathbf{f})|^2 = (\mathbf{f}, D^\beta \mathbf{f}) \qquad (5.60)$$

This is a trace of the famous *Higgs mechanism* (in solid state physics discovered by Anderson. For related results see Ref. 4).

(III) As a generalization of Theorem 5.5, Section 5.2, we have the following result:

Theorem 5.8 (Existence of the thermodynamic limit in nonrelativistic QED). For charge-conjugation-invariant systems,

$$\rho_{\Lambda,1}(\beta, z; (x)_N(x')_M; (y)_N(y')_M; \mathbf{f})$$

defined in (5.55) is monotone increasing in Λ and z and decreasing when D^β increases, in the quadratic form sense. In particular, the limiting RDMs

$$\rho_1(\beta, z;...) = \lim_{\Lambda \nearrow \mathbb{R}^\nu} \rho_{\Lambda,1}(\beta, z;...)$$

exist if $z < \exp(-\beta q^2 K/2)$.

Proof. By (5.55), uniform upper bounds on $\rho_{\Lambda,1}(\beta, z; (x)_N, ..., (y')_M; \mathrm{f})$ follow directly from Theorem 5.7(i). Theorem 5.1, and the trivial inequality

$$|\{\exp[i\mathbf{A}(\mathbf{f})]\}\Xi_{\Lambda,1}(\beta, z; \mathbf{A})| \leqslant \Xi_{\Lambda,1}(\beta, z; \mathbf{A})$$

The proofs of monotonicity in Λ, z, and D^β are the same as those of Theorems 5.5 and 5.6 if one uses, instead of Theorem 3.1(ii), Theorem 3.3. ■

More details concerning a related result may be found in Ref. 4.

6. SOME OPEN PROBLEMS AND OUTLOOK

The following five topics may be worth being studied within the functional integral formalism developed in this paper.

1. Behavior at small values of z and β, decay of correlations in the thermodynamic limit, cluster expansion,[3,12] screening properties.[2]

2. Analysis of phase diagram based on studying the behavior of the "action" $S_{\Lambda,\epsilon}(m, z; \phi)$ [resp. $\mathbb{C}_{\Lambda,\epsilon}(\beta, z; q\phi)$; see Section 4.2] as a functional of ϕ. Reliability of the naive Goldstone picture with ϕ as order parameter.

3. Further study of Bose–Einstein condensation (e.g., for charge-conjugation invariant systems), in continuation of the results given in Theorems 5.3 and 5.4.

4. Continuation of analysis of nonrelativistic (quantum) electrodynamics and superconductivity for nonrelativistic bosons.

5. Existence of the classical limit ($\hbar \searrow 0$) of RDMs and other correlation functions.

We conclude with a few comments on some of these circles of problems.

1. The functional integral formalism developed in this paper would in principle enable one to apply the Glimm–Jaffe–Spencer cluster expansion[26] to the quantum mechanical gases considered in this paper, provided β and z are suitably small, and the potential V is of rapid decrease.

This may improve the results of Ginibre[12] and simplify the techniques of Brydges and Federbush,[3] but one cannot expect that the results of Brydges and Federbush[3] can be improved in this way. (We notice that the applicability of the cluster expansion does *not* require charge-conjugation invariance.)

More interesting is the question of whether quantum mechanical gases of particles interacting via regularized Coulomb potentials will have Debye screening[2] for tiny values of β. In principle, a combination of the methods developed in this paper and in Ref. 2 ought to yield insight into this problem.

2. One can imagine that one may extend the Glimm–Jaffe–Spencer version of the Peierls argument[22] and their mean-field contour expansion[27] to the systems considered in this paper, by viewing the auxiliary random field

ϕ as an order parameter (the analog of the Ising spin in the conventional Peierls argument). Related to this is the discussion of the properties of

$$S(\phi) \equiv \lim_{\Lambda \nearrow \mathbb{R}^\nu} \frac{1}{|\Lambda|} S_\Lambda(\phi) \tag{6.1}$$

$$S_\Lambda(\phi) = \begin{cases} -S_{\Lambda,\epsilon}(m, z; \phi) + \frac{1}{2}(\phi, \mathbb{V}_\Lambda^{-1}\phi) & (6.2) \\ \text{or} \\ -\mathbb{C}_{\Lambda,\epsilon}(\beta, z; q\phi) + \frac{1}{2}(\phi, \mathbb{V}_\Lambda^{-1}\phi) & (6.3) \end{cases}$$

for fields ϕ which are constant on $\mathbb{R}^\nu \times [0, \beta]$. (The functionals $S_{\Lambda,\epsilon}$ and $\mathbb{C}_{\Lambda,\epsilon}$ are defined in Section 4.2.) This supplies an analog of the Goldstone picture. For S_Λ as in (6.3) and a translation-invariant potential V we obtain

$$S(\phi) = -\left(\frac{2\pi m}{\beta}\right)^{\nu/2} \sum_{j=1}^{\infty} \frac{\epsilon^{j-1}}{j^{1+\nu/2}} \times \left\{z \exp\left[\frac{\beta}{2} q^2 V(0)\right]\right\}^j \cos(j\beta q\phi) + \frac{\beta}{2} \hat{\mathbb{V}}(0)^{-1}\phi^2 \tag{6.4}$$

Heuristically, one expects that if $S(0)$ is the unique global minimum of $S(\phi)$, the infinite-volume equilibrium state is unique. If there are degenerate absolute minima for some values of z and $\beta(\epsilon = -1)$, then the equilibrium state is presumably not extremal (i.e., degenerate). This situation is met in a system of fermions on a lattice ($\mathbb{R}^\nu \to \mathbb{Z}^\nu$) with *attractive* interactions, resp. in the quantum mechanical Widom–Rowlinson model on the lattice, with Fermi statistics. The combination of a lattice with Fermi statistics renders such systems *stable*. The functional integral formalism for lattice systems with attractive potentials is obtained from the one developed in Section 4 by using Brownian motion on the lattice and replacing

$$\exp\left[iq \int_0^\beta d\tau\, \phi(\omega(+k\beta), \tau)\right]$$

by

$$\exp\left[q \int_0^\beta d\tau\, \phi(\omega(\tau + k\beta, \tau)\right]$$

For $\epsilon = -1$ and strictly positive lattice spacing the resulting expressions make sense. The formalism may be useful to develop a microscopic theory of Cooper pair formation for lattice electrons.

3. Nonrelativistic quantum electrodynamics is a subject that has been undeservedly neglected. Most problems one may wish to pose are still open.

(a) Do atoms coupled to the (ultraviolet regularized) quantized radiation field have discrete ground states? What is the correct mathematical description of the resonances corresponding to the excited, atomic states?

(b) Do nonrelativistic, interacting Bose gases exhibit the Meissner effect typical of a superconducting state, at suitable density and temperature? Do such systems exhibit the formation of vortices? Do nonrelativistic, interacting Bose gases coupled to the quantized radiation field exhibit the Higgs mechanism in a strict sense of the word?

4. For bounded regions Λ the existence of the classical limit can be proven for the RDMs of the system analyzed in Sections 4 and 5.1. If one sets $z_\hbar = z(2\pi\beta\hbar)^{\nu/2}$ and if one replaces Δ by $\hbar^2\Delta$, the RDMs converge to the corresponding classical correlation functions studied in paper I. If one uses the functional integral formalism of Section 4 and appropriate L^p estimates the proof is particularly straightforward. The exchange of $\Lambda \nearrow \mathbb{R}^\nu$ and $\hbar \searrow 0$ is, however, nontrivial (one could use a cluster expansion).

APPENDIX A. PROOF OF THEOREM 2.1

Let

$$U_l((x)_M, (x')_N) = \sum_{1 \leqslant i<j \leqslant M} q_1{}^2 V_l(x_i, x_j) + \sum_{1 \leqslant i<j \leqslant N} q_2{}^2 V_l(x_i', x_j') + \sum_{i=1}^{M} \sum_{j=1}^{N} q_1 q_2 V_l(x_i, x_j) \tag{A.1}$$

for $l = 1, 2$.

Recall that $V = V_1 + V_2$, where $V_2(x, y) \equiv V_2(x - y)$ is a function with nonnegative, continuous Fourier transform $\hat{V}_2$, and

$$\hat{V}_2(0) > 0 \tag{A.2}$$

Without loss of generality we may assume that

$$q_2 > 0 \tag{A.3}$$

Finally we recall that the statistics of the first species of particles is Fermi statistics, i.e.,

$$\epsilon_1 = -1 \tag{A.4}$$

but $\epsilon_2 = \pm 1$.

By inequality (2.1) and hypothesis (2.3)

$$H_\Lambda^{(M,N)} \geqslant \tilde{H}_\Lambda^{(M,N)} - B(M + N) \qquad \text{with } \tilde{H}_\Lambda^{(M,N)} \equiv \tfrac{1}{2} T_\Lambda^{(M,N)} + U_2((x)_M, (x')_N) \tag{A.5}$$

Since $\hat{V}_2(k) \geqslant 0$, there exists a finite constant $\tilde{B}$ $[=\max(q_1{}^2, q_2{}^2)\cdot V(0)]$ such that

$$U_2((x)_M, (x')_N) \geqslant -\tilde{B}(M + N) \tag{A.6}$$

(see Ref. 22). For the proof of Theorem 2.1 we need the following result:

Lemma A.1 Suppose that $q_2 N \geqslant 2|q_1|M$, and $x_j \in \Lambda, x_i' \in \Lambda$, for $j = 1,\ldots, M$ and $i = 1,\ldots, N$. Then

$$U_2((x)_M, (x')_N) \geqslant c_1 N^2/|\Lambda| - c_2 N$$

for some finite constants $c_1 > 0$ and c_2.

Proof. We use some arguments due to Ruelle.[23] Let

$$n_1(x) = \sum_{j=1}^{M} q_1\, \delta(x - x_j), \qquad n_2(x) = \sum_{i=1}^{N} q_2\, \delta(x - x_i')$$

with $x_j \in \Lambda$, $x_i' \in \Lambda$ for $j = 1,\ldots, M$ and $i = 1,\ldots, N$. Let

$$\hat{n}_l(p) = (2\pi)^{-\nu/2} \int_\Lambda n_l(x) \exp(-ip\cdot x) d^\nu x) = (2\pi)^{-\nu/2} \sum_k q_l \exp(ip\cdot x_k)$$

Clearly

$$\begin{aligned} U_2((x)_M, (x')_N) &= \int_\Lambda d^\nu x \int_\Lambda d^\nu y [n_1(x) + n_2(x)] V_2(x-y) [n_1(y) + n_2(y)] \\ &\quad - q_1{}^2 V(0) M - q_2{}^2 V(0) N \\ &= \int d^\nu p\ \hat{V}_2(p) |\hat{n}_1(p) + \hat{n}_2(p)|^2 - q_1{}^2 V(0) M - q_2{}^2 V(0) N \end{aligned} \tag{A.7}$$

Next

$$\begin{aligned} |\hat{n}_1(p) + \hat{n}_2(p)|^2 &= (2\pi)^{-\nu/2} \int_\Lambda d^\nu x\, e^{-ip\cdot x} \\ &\quad \times \int_\Lambda d^\nu y\, [n_1(x+y) + n_2(x+y)][n_1(y) + n_2(y)] \end{aligned}$$

By power series expansion of $e^{ip\cdot x}$ and the hypothesis that Λ be regular, i.e.,

$$\max_{x\in\Lambda} |x| \leqslant \alpha |\Lambda|^{1/\nu} \qquad \text{for some finite } \alpha$$

provided $\Lambda \ni 0$ (which can be assumed due to the translation of V_2), we have

$$|\hat{n}_1(p) + \hat{n}_2(p)|^2 \geqslant \max(0, G(p))$$

with

$$\begin{aligned} G(p) &= (2\pi)^{-\nu/2} \left\{ \int_\Lambda d^\nu x\, [n_1(x) + n_2(x)] \right\}^2 \\ &\quad - (2\pi)^{-\nu/2} \left\{ \int_\Lambda d^\nu x\, [|n_1(x)| + |n_2(x)|] \right\}^2 [\exp(\alpha |p| |\Lambda|^{1/\nu}) - 1] \end{aligned} \tag{A.8}$$

Since by hypothesis $q_2 N \geqslant 2|q_1|M$,

$$\begin{aligned} \int d^\nu x\, [n_1(x) + n_2(x)] &= q_1 M + q_2 N \geqslant \tfrac{1}{2} q_2 N \\ \int d^\nu x\, [|n_1(x)| + |n_2(x)|] &= |q_1| M + q_2 N \leqslant \tfrac{3}{2} q_2 N \end{aligned} \tag{A.9}$$

Let

$$f(p) \equiv \max(0, 1 - 9[e^{\alpha|p|} - 1]) \tag{A.10}$$

Clearly $f(p)$ is a nonnegative, continuous function with

$$\operatorname{supp} f = \{p : |p| \leqslant \alpha^{-1} \ln(10/9)\}$$

compact.

By (A.8) and (A.9)

$$\begin{aligned} |\hat{n}_1(p) + \hat{n}_2(p)|^2 &\geqslant (2\pi)^{-\nu/2} \tfrac{1}{4} (q_2 N)^2 \max(0, 1 - 9[\exp(\alpha|p||\Lambda|^{1/\nu}) - 1]) \\ &= (2\pi)^{-\nu/2} \tfrac{1}{4} (q_2 N)^2 f(|\Lambda|^{1/\nu} p) \end{aligned} \tag{A.11}$$

Thus, using (A.7) and the inequality

$$q_1{}^2 V(0) M + q_2{}^2 V(0) N \leqslant q_2{}^2 V(0)(1 + |q_1|/2q_2) N$$

which holds, since $|q_1| M < (q_2/2) N$ by hypothesis, we obtain

$$\begin{aligned} U_2((x)_M, (x')_N) &\geqslant (2\pi)^{-\nu/2} \frac{1}{4} (q_2 N)^2 \int d^\nu p\, \hat{V}_2(p) f(|\Lambda|^{1/\nu} p) - c_2 N \\ &= (2\pi)^{-\nu/2} \frac{q_2{}^2}{4} \frac{N^2}{|\Lambda|} \int d^\nu k\, \hat{V}_2(|\Lambda|^{-1/\nu} k) f(k) - c_2 N \\ &\geqslant c_1 \frac{N^2}{|\Lambda|} - c_2 N \end{aligned}$$

with $c_2 = q_2{}^2 V(0)(1 + |q_1|/2q_2)$ and

$$c_1 = (2\pi)^{-\nu/2} (q_2{}^2/4) \min_{|k| \leqslant \alpha^{-1} \ln(10/9)} \hat{V}(k|\Lambda|^{-1/\nu}) \int d^\nu k\, f(k)$$

which is strictly positive if $|\Lambda|$ is sufficiently large, because

$$\lim_{|\Lambda| \nearrow \infty} \hat{V}(k|\Lambda|^{-1/\nu}) = \hat{V}(0) > 0 \qquad \text{for } |k| < \infty$$

and $\int d^\nu k\, f(k) > 0$ by (A.10). ∎

We are now in a position to prove Theorem 2.1. By (A.5) and the definition of $\Xi_\Lambda(\beta, z_1, z_2)$ [see (1.6)]

$$\Xi_\Lambda(\beta, z_1, z_2) \leqslant \sum_{M=0}^{\infty} \sum_{N=0}^{\infty} \bar{z}_1{}^M \bar{z}_2{}^N \operatorname{Tr}_{\mathscr{H}_\Lambda^{(M,N)}}\{\exp[-\beta \tilde{H}_\Lambda^{(M,N)}]\} \tag{A.12}$$

where $\bar{z}_l = z_l \exp(\beta B)$, $l = 1, 2$.

Next,

$$\sum_{N=0}^{\infty} \cdot = \sum_{N=0}^{\tilde{N}} \cdot + \sum_{N=\tilde{N}+1}^{\infty} \qquad \text{with } \tilde{N} \equiv [2|q_1|M/q_2] \tag{A.13}$$

where $[a]$ is the largest integer $\leqslant a$.

By (A.6),

$$\begin{aligned}
&\sum_{N=0}^{\tilde{N}} \bar{z}_2{}^N \operatorname{Tr}_{\mathscr{H}_\Lambda^{(M,N)}}\{\exp(-\beta \tilde{H}_\Lambda^{(M,N)})\} \\
&\quad \leqslant d_1{}^M \operatorname{Tr}_{\mathscr{H}_\Lambda^{(M,0)}}\left\{\exp\left(-\frac{\beta}{2} T_\Lambda^{(M,0)}\right)\right\} \sum_{N=0}^{\tilde{N}} d_2{}^N \operatorname{Tr}_{\mathscr{H}_\Lambda^{(0,N)}}\left\{\exp\left(-\frac{\beta}{2} T_\Lambda^{(0,N)}\right)\right\} \\
&\quad \leqslant \{d_1(2d_2)^{2|q_1|/q_2}\}^M \operatorname{Tr}_{\mathscr{H}_\Lambda^{(M,0)}}\left\{\exp\left(-\frac{\beta}{2} T_\Lambda^{(M,0)}\right)\right\} \exp(d_3|\Lambda|)
\end{aligned} \tag{A.14}$$

where $d_1 = \exp(\beta \tilde{B})$ and $d_2 = \max(\frac{1}{2}, \bar{z}_2 \exp(\beta \tilde{B}))$, and we have used the inequalities

$$\begin{aligned}
&\sum_{N=0}^{\tilde{N}} d_2{}^N \operatorname{Tr}_{\mathscr{H}_\Lambda^{(0,N)}}\left\{\exp\left(-\frac{\beta}{2} T_\Lambda^{(0,N)}\right)\right\} \\
&\quad \leqslant (2d_2)^{\tilde{N}} \sum_{N=0}^{\infty} (\tfrac{1}{2})^N \operatorname{Tr}_{\mathscr{H}_\Lambda^{(0,N)}}\left\{\exp\left(-\frac{\beta}{2} T_\Lambda^{(0,N)}\right)\right\} \\
&\quad \leqslant (2d_2)^{2|q_1|M/q_2} \exp(d_3|\Lambda|)
\end{aligned}$$

for some $d_3 < \infty$, for both $\epsilon_2 = -1$ and $\epsilon_2 = +1$.

Next we apply Lemma A.1 to obtain

$$\begin{aligned}
&\sum_{N=\tilde{N}+1}^{\infty} \bar{z}_2{}^N \operatorname{Tr}_{\mathscr{H}_\Lambda^{(M,N)}}\{\exp(-\beta \tilde{H}_\Lambda^{(M,N)})\} \\
&\quad \leqslant \operatorname{Tr}_{\mathscr{H}_\Lambda^{(M,0)}}\{\exp(-\tfrac{1}{2}\beta T_\Lambda^{(M,0)})\} \\
&\quad\quad \times \left\{ \sum_{\{N: q_2 N \geqslant 2|q_1|M\}} \bar{z}_2{}^N \exp(c_2 N) \exp(-c_1 N^2/|\Lambda|) \right. \\
&\quad\quad \left. \times \operatorname{Tr}_{\mathscr{H}_\Lambda^{(0,N)}}\{exp(-\tfrac{1}{2}\beta T^{(0,N)})]\right\}
\end{aligned}$$

Now, choose $\gamma \geqslant 1$ to be so large that

$$\bar{z}_2 e^{c_2} e^{-\gamma c_1} < 1/2 \tag{A.15}$$

In that case,

$$\begin{aligned}
&\sum_{q_2 N \geqslant 2|q_1|M} \bar{z}_2{}^N \exp(c_2 N) \exp(-c_1 N^2/|\Lambda|) \operatorname{Tr}_{\mathscr{H}_\Lambda^{(0,N)}}\{\exp(-\tfrac{1}{2}\beta T_\Lambda^{(0,N)})\} \\
&\leqslant \sum_{N=0}^{[\gamma|\Lambda|]} \bar{z}_2{}^N \exp(c_2 N) \operatorname{Tr}_{\mathscr{H}_\Lambda^{(0,N)}}\{\exp(-\tfrac{1}{2}\beta T_\Lambda^{(0,N)})\} \\
&\quad + \sum_{N=[\gamma|\Lambda|]+1}^{\infty} (\tfrac{1}{2})^N \operatorname{Tr}_{\mathscr{H}_\Lambda^{(0,N)}}\{\exp(-\tfrac{1}{2}\beta T_\Lambda^{(0,N)})\} \\
&= \left(\sum_{N=0}^{[\gamma|\Lambda|]} \{2\bar{z}_2 \exp(c_2)\}^N + 1 \right) \sum_{N=0}^{\infty} (\tfrac{1}{2})^N \operatorname{Tr}_{\mathscr{H}_\Lambda^{(0,N)}}\{\exp(-\tfrac{1}{2} T_\Lambda^{(0,N)})\} \\
&\leqslant 2(d_4 - 1)^{-1} d_4^{\gamma|\Lambda|} \exp(d_3|\Lambda|)
\end{aligned} \tag{A.16}$$

with $d_4 = \max(2, 2\bar{z}_2 e^{c_2})$, and we have used (A.15), the Schwarz inequality for series, the inequality $(\sum a_n{}^2)^{1/2} \leqslant \sum a_n$, with $a_n \geqslant 0$, in that order.

By (A.12), (A.14), and (A.16)

$$\begin{aligned}
\Xi_\Lambda(\beta, z_1, z_2) \leqslant{}& 4(d_4 - 1)^{-1} d_4^{\gamma|\Lambda|} \exp(d_3|\Lambda|) \\
&\times \sum_{M=0}^{\infty} \bar{z}^M \operatorname{Tr}_{\mathscr{H}_\Lambda^{(M,0)}}\{\exp(-\tfrac{1}{2}\beta T_\Lambda^{(M,0)})\}
\end{aligned}$$

where $\bar{z} = \max(\bar{z}_1, \bar{z}_1 d_1 (2d_2)^{2|q_1|/q_2})$.

Finally, since all vectors in $\mathscr{H}_\Lambda{}^{(M,0)}$ have Fermi statistics,

$$\sum_{M=0}^{\infty} \bar{z}^M \operatorname{Tr}_{\mathscr{H}_\Lambda^{(M,0)}}\{\exp(-\tfrac{1}{2}\beta T_\Lambda^{(M,0)})\} \leqslant \exp(d_5|\Lambda|)$$

for some constant d_5 which is finite when $\bar{z} < \infty$. This completes the proof of Theorem 2.1. ■

APPENDIX B. PROOF OF THEOREM 2.2

In Theorem 2.2, $q_1 > 0$, $q_2 < 0$, $\epsilon_1 = \epsilon_2 = 1$ (i.e., both species have Bose statistics), $z_1 > 0$, $z_2 > 0$. One sets

$$\begin{aligned}
m &= \min\{m_1, m_2\} \\
z &= \tfrac{1}{2} \min\{z_1, z_2\} \min\{1 - q_1/q_2, 1 - q_2/q_1\}
\end{aligned} \tag{B.1}$$

[see (2.5)]. Let

$$\tilde{T}_\Lambda^{(M+N)} = -\sum_{i=1}^{M} \frac{1}{2m} \Delta_i{}^\Lambda - \sum_{j=1}^{N} \frac{1}{2m} \Delta_j'^\Lambda \tag{B.2}$$

$$\tilde{H}_\Lambda^{(M,N)} = \tilde{T}_\Lambda^{(M+N)} + U((x)_M, (x')_N) \tag{B.3}$$

By (B.1),

$$H_\Lambda^{(M,N)} \leqslant \tilde{H}_\Lambda^{(M,N)} \tag{B.4}$$

We define

$$\tilde{\mathscr{H}}_\Lambda^{(M+N)} = L^2(\Lambda, d^\nu x)^{\otimes_1 (M+N)} \otimes \mathbb{C} \tag{B.5}$$

The second factor stands for the spin wave function that is an eigenvector of the 1-component of the total spin operator with maximal eigenvalue, for example. Obviously

$$\tilde{\mathscr{H}}_\Lambda^{(M+N)} \subset \mathscr{H}_\Lambda^{(M,N)} \tag{B.6}$$

with $\mathscr{H}_\Lambda^{(M,N)}$ given by (1.4), (1.4′). By (B.4) and (B.6),

$$\mathrm{Tr}_{\mathscr{H}_\Lambda^{(M,N)}}[\exp(-\beta H_\Lambda^{(M,N)})] \geqslant \mathrm{Tr}_{\tilde{\mathscr{H}}_\Lambda^{(M+N)}}[\exp(-\beta \tilde{H}_\Lambda^{(M,N)})] \tag{B.7}$$

Let $\bar{z} = \min\{z_1, z_2\}$. By definition of $\Xi_\Lambda(\beta, z_1, z_2)$ and (B.7),

$$\begin{aligned}
\Xi_\Lambda(\beta, z_1, z_2) &\geqslant \sum_{M,N=0}^{\infty} \bar{z}^{M+N} \mathrm{Tr}_{\tilde{\mathscr{H}}_\Lambda^{(M+N)}}[\exp(-\beta \tilde{H}_\Lambda^{(M,N)})] \\
&= \sum_{K=0}^{\infty} \bar{z}^K \sum_{N=0}^{K} \mathrm{Tr}_{\tilde{\mathscr{H}}_\Lambda^{(K)}}[\exp(-\beta \tilde{H}_\Lambda^{(K-N,N)})] \qquad \text{(B.8)} \\
&\geqslant \sum_{K=0}^{\infty} (\bar{z}/2)^K \sum_{N=0}^{K} \binom{K}{N} \mathrm{Tr}_{\tilde{\mathscr{H}}_\Lambda^{(K)}}[\exp(-\beta \tilde{H}_\Lambda^{(K-N,N)})] \\
&= \sum_{K=0}^{\infty} (\bar{z}/2)^K \int \prod_{j=1}^{K} [\delta(\rho_j - q_1) + \delta(\rho_j - q_2)]\, d\rho_j \\
&\quad \times \mathrm{Tr}_{\tilde{\mathscr{H}}_\Lambda^{(K)}}\{\exp[-\beta \tilde{H}_\Lambda{}^K(\{\rho\})]\} \qquad \text{(B.9)}
\end{aligned}$$

where

$$H_\Lambda{}^K(\{\rho\}) = \tilde{T}_\Lambda^{(K)} + U((x)_K, \{\rho\}) \tag{B.10}$$

with

$$U((x)_K, \{\rho\}) = \sum_{1 \leqslant i < j \leqslant K} \rho_i \rho_j V(x_i, x_j)$$

Without loss of generality we may assume $q_1 \geqslant |q_2|$. From (B.8) and (B.9) we now obtain

$$\Xi_\Lambda(\beta, z_1, z_2) \geqslant \sum_{K=0}^{\infty} z^K \int \prod_{i=1}^{K} d\lambda(\rho_i) \operatorname{Tr}_{\mathscr{H}_\Lambda^{(K)}}\{\exp[-\beta \tilde{H}^K(\{\rho\})]\} \tag{B.11}$$

where $z = \frac{1}{2}\bar{z}(1 + |q_2|/q_1)$, and

$$d\lambda(\rho) = (1 + |q_2|/q_1)^{-1}[|q_2|/q_1\, \delta(\rho - q_1) + \delta(\rho - q_2)]\, d\rho \tag{B.12}$$

In (B.11) we have used that

$$(1 + |q_2|/q_1)\, d\lambda(\rho) \leqslant [\delta(\rho - q_1) + \delta(\rho - q_2)]\, d\rho$$

We now express

$$\operatorname{Tr}_{\mathscr{H}^{(K)}}\{\exp[-\beta \tilde{H}_\Lambda^{(K)}(\{\rho\})]\}$$

in terms of a Feynman–Kac integral (see Section 4.1). Then we apply the Jensen–Symanzik inequality with respect to the Wiener measure. Subsequently one may "undo" the Feynman–Kac integrals. This yields

$$\begin{aligned} &\operatorname{Tr}_{\mathscr{H}_\Lambda^{(K)}}\{\exp[-\beta \tilde{H}_\Lambda^{(K)}(\{\rho\})]\} \\ &\qquad \geqslant \operatorname{Tr}_{\mathscr{H}_\Lambda^{(K)}}\{\exp[-\beta \tilde{T}_\Lambda^{(K)}]\} \exp[-\beta \langle U((\cdot)_K, \{\rho\})\rangle_0] \end{aligned} \tag{B.13}$$

with

$$\langle - \rangle_0 = \{\operatorname{Tr}_{\mathscr{H}_\Lambda^{(K)}}[\exp(-\beta \tilde{T}_\Lambda^{(K)})]\}^{-1} \operatorname{Tr}_{\mathscr{H}_\Lambda^{(K)}}[\exp(-\beta \tilde{T}_\Lambda^{(K)}) -]$$

Next, we apply Jensen's inequality with respect to $\int \prod_{i=1}^{K} d\lambda(\rho_i)$–, using the fact that $\int d\lambda(\rho) = 1$. This yields

$$\begin{aligned} \Xi_\Lambda(\beta, z_1, z_2) \geqslant \sum_{K=0}^{\infty} & z^K \operatorname{Tr}_{\mathscr{H}_\Lambda^{(K)}}[\exp(-\beta \tilde{T}_\Lambda^{(K)})] \\ & \times \left\{\exp\left[-\beta \int \prod_{i=1}^{K} d\lambda(\rho_i) \langle U((\cdot)_K, \{\rho\})\rangle_0 \right]\right\} \end{aligned}$$

Finally,

$$\begin{aligned} \int \prod_{i=1}^{K} d\lambda(\rho_i) \langle U((\cdot)_K, \{\rho\})\rangle_0 &= \tfrac{1}{2}N(N-1) \int d\lambda(\rho)\, \rho \int d\lambda(\rho')\, \rho' \langle V(\cdot, \cdot)\rangle_0 \\ &= 0 \qquad \text{by definition (B.12) of } d\lambda \end{aligned}$$

This completes the proof of Theorem 2.2. ■

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